THE COLLEGE OF WILLIAM AND MARY

Phase-space methods in plasma wave theory

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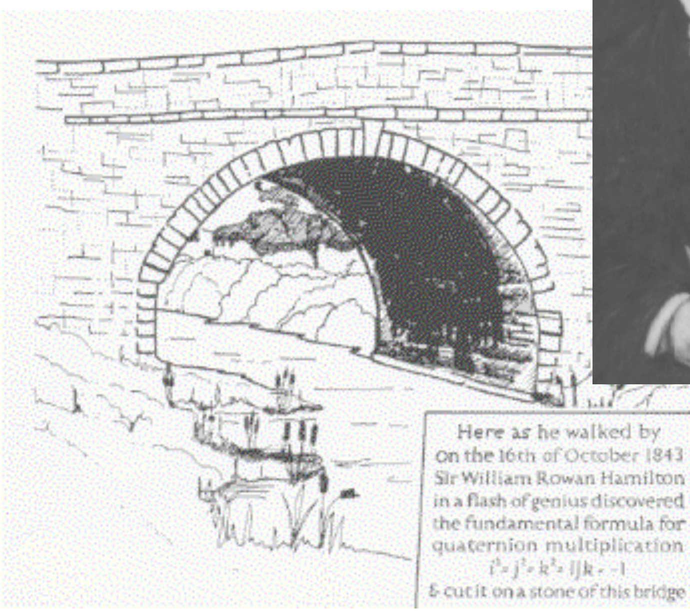
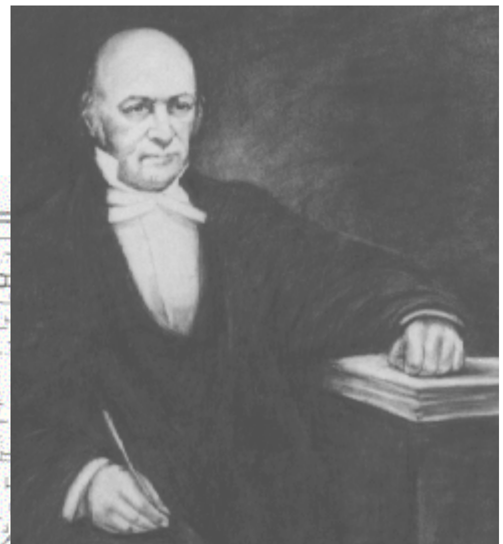
Outline:

- A little history and background
- Vector WKB: a good approximation
- When good approximations good bad: caustics
- When good approximations good bad: mode conversion
- Applications
- The future





In the beginning there was Hamilton…

And Hamilton said "Let there be rays."

www.hamilton.tcd.ie/events/gt/gt.htm

Themes

- Geometrical pictures give insight.
- Invariant formulations and symmetries are important.
- Modern mathematics and physics have much to learn from one another.
- General formulations will have wide applicability.

Plasma preliminaries…

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0.$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}.$$

Maxwell’s eqs.

Model for the matter: e.g. current carrying fluid, kinetic model (Vlasov), etc.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

$$\mathbf{j}(x,t) = \int_{-\infty}^{+\infty} dt' \Theta(t-t') \int d^2x \sigma(\mathbf{x},\mathbf{x}',t-t') \cdot \mathbf{E}(\mathbf{x}',t'), \quad sim.\ \rho[\mathbf{E}]$$

Plasma preliminaries…

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0.$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}.$$

Maxwell's eqs.

Model for the matter: e.g. current carrying fluid, kinetic model (Vlasov), etc.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

In the following, we assume *linear* responses

$$\mathbf{j}(x,t) = \int_{-\infty}^{+\infty} dt' \Theta(t-t') \int d^2x \sigma(\mathbf{x},\mathbf{x}',t-t') \cdot \mathbf{E}(\mathbf{x}',t'), \quad sim.\ \rho[\mathbf{E}]$$

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi}{c^2}\int \Theta(t-t')\frac{\partial \sigma(\mathbf{x},\mathbf{x}',t-t')}{\partial t}\cdot \mathbf{E}(\mathbf{x}',t') = 0.$$

Now use the identity:

$$\frac{\partial^2 \phi(\mathbf{x},t)}{\partial t^2} = \int dt' d^2x' \delta(t-t')\delta(x_1 - x_1')\delta(x_2 - x_2')\frac{\partial^2 \phi(\mathbf{x}',t')}{\partial t'^2}.$$

(*sim.* for gradients) to cast into standard form:

$$\boxed{\int d^2x'dt'\, \mathbf{D}(\,\mathbf{x},\mathbf{x}',t-t'\,)\cdot \mathbf{E}(\,\mathbf{x}',t'\,) = 0.}$$

Vector WKB (time-stationary, conservative)

$$\int d^2x' dt' \mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \cdot \mathbf{E}(\mathbf{x}', t') = 0.$$

Dispersion matrix
(Weyl symbol)
A hermitian matrix
for all *real*
(x,k,ω**)**

$$D_{mn}(\mathbf{x}, \mathbf{k}, \omega) \equiv \int d\mathbf{s} d\tau \, e^{i(\mathbf{k}\cdot\mathbf{s} - \omega\tau)} D_{mn}\left(\mathbf{x} + \frac{\mathbf{s}}{2}, \mathbf{x} - \frac{\mathbf{s}}{2}; \boldsymbol{\tau}\right)$$

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \mathbf{E}(\mathbf{x}, t) = 0.$$

$$\int d^2x' dt' \mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \cdot \mathbf{E}(\mathbf{x}', t') = 0.$$

If this is non-local

The dispersion function (Weyl symbol) will usually be smooth in its arguments.

$$\boxed{\mathbf{D}(\mathbf{x}, \mathbf{k}, \omega)}$$

This will be infinite order (pseudo-differential)

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \mathbf{E}(\mathbf{x}, t) = 0.$$

A theme from mathematics: symbols of operators (H. Weyl, 1930's)

$$\hat{A}$$

$$\Downarrow$$

$$\langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle = A(\mathbf{x}, \mathbf{x}')$$

$$\Downarrow$$

$$\Sigma_A(\mathbf{x}, \mathbf{k})$$

Invariants, geometry, harmonic analysis (i.e. fourier analysis on groups…

A theme from mathematics: symbols of operators (H. Weyl, 1930's)

$$\hat{A}$$

An abstract operator acting in some Hilbert space

$$\downarrow$$

$$\langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle = A(\mathbf{x}, \mathbf{x}')$$

A matrix representation of the operator in some particular basis

$$\downarrow$$

$$\Sigma_A(\mathbf{x}, \mathbf{k})$$

The symbol of the operator (a function on phase space)

A theme from mathematics: symbols of operators (H. Weyl, 1930's)

$$\hat{A}$$

An abstract operator acting in some Hilbert space

$$\langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle = A(\mathbf{x}, \mathbf{x}')$$

A matrix representation of the operator in some particular basis

$$\Sigma_A(\mathbf{x}, \mathbf{k})$$

The symbol of the operator (a function on phase space)

Symbols of operators

$$\hat{A}$$

$$\Updownarrow$$

$$\Sigma_A(\mathbf{x},\mathbf{k})$$

This mapping is a type of fourier transform. (N. Zobin)

Ordinary fourier transforms are associated with the (commutative) translation group in **x**.

This new fourier transform is associated with (non-commutative) shifts on phase space, **(x,k)** (the Heisenberg-Weyl group)

Example of a symbol: the Wigner function

$$\psi(\mathbf{x}) \quad \Rightarrow \quad \underbrace{W(\mathbf{x},\mathbf{k})}_{a\,\text{'density' on phase space}} = \int d^n s\, e^{i\mathbf{k}\cdot\mathbf{s}} \psi^*\left(x + \frac{s}{2}\right) \psi\left(x - \frac{s}{2}\right)$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$\langle \mathbf{x} | \hat{\rho} | \mathbf{x}' \rangle = \psi^*(\mathbf{x}) \psi(\mathbf{x}')$$

$$W(\mathbf{x},\mathbf{k})$$

Symbols of products of operators

Moyal - 1930's (physics)
Algebra deformations - current mathematics

A convolution on the Heisenberg-Weyl group: the 'Moyal product'

Poisson bracket

$$\Sigma_{AB}(\mathbf{x},\mathbf{k}) = \Sigma_A(\mathbf{x},\mathbf{k}) * \Sigma_B(\mathbf{x},\mathbf{k})$$

$$= \Sigma_A(\mathbf{x},\mathbf{k})\Sigma_B(\mathbf{x},\mathbf{k}) - \frac{i}{2}\left\{\Sigma_A(\mathbf{x},\mathbf{k}),\Sigma_B(\mathbf{x},\mathbf{k})\right\} + \ldots$$

Back to vector WKB…

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \mathbf{E}(\mathbf{x}, t) = 0.$$

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \left[e^{i(\theta(\mathbf{x}) - \omega t)} E(\mathbf{x}) \hat{\mathbf{e}}(\mathbf{x}) \right] = 0.$$ Eikonal ansatz

$$e^{i(\theta(\mathbf{x}) - \omega t)} \mathbf{D}(\mathbf{x}, \nabla\theta - i\nabla, \omega) \cdot \left[E(\mathbf{x}) \hat{\mathbf{e}}(\mathbf{x}) \right] = 0.$$

$$e^{i(\theta(\mathbf{x}) - \omega t)} \left[\mathbf{D}(\mathbf{x}, \nabla\theta, \omega) - i \underbrace{\nabla_k \mathbf{D} \cdot \nabla}_{symmetrized!} + \ldots \right] \cdot \left[E(\mathbf{x}) \hat{\mathbf{e}}(\mathbf{x}) \right] = 0.$$

$$\left[\mathbf{D}(\mathbf{x},\nabla\theta,\omega) - i\underbrace{\nabla_k \mathbf{D}\cdot\nabla}_{symmetrized!} + \ldots\right]\cdot\left[E(\mathbf{x})\hat{\mathbf{e}}(\mathbf{x})\right] = 0.$$

Leading order implies

$$\mathbf{D}(\mathbf{x},\nabla\theta,\omega)\cdot\hat{\mathbf{e}}(\mathbf{x}) = 0.$$

1. $\mathbf{D}(\mathbf{x},\nabla\theta(\mathbf{x}),\omega)$ *must have a null eigenvalue at* $\mathbf{x}$.
2. The polarization must be the associated null eigenvector.

The null eigenvalue condition implies

$$\det \mathbf{D}(\mathbf{x}, \nabla\theta, \omega) \equiv D(\mathbf{x}, \nabla\theta, \omega) = 0. \quad \text{“\textit{Hamilton-Jacobi}”}$$

This condition is a nonlinear PDE for the unknown $\theta(\mathbf{x})$.
How to solve it? Hamilton showed the way…

$$D(\mathbf{x}, \mathbf{k} = \nabla\theta, \omega) = 0.$$

Given $\theta(\mathbf{x})$, this holds along *any* curve $\mathbf{x}(\sigma)$.

$$D(\mathbf{x}(\sigma), \mathbf{k}(\sigma), \omega) = 0.$$

If we choose the particular curve satisfying $\dfrac{d\mathbf{x}}{d\sigma} = -\nabla_k D$

Then: $\dfrac{d\mathbf{k}}{d\sigma} = \nabla_x D$

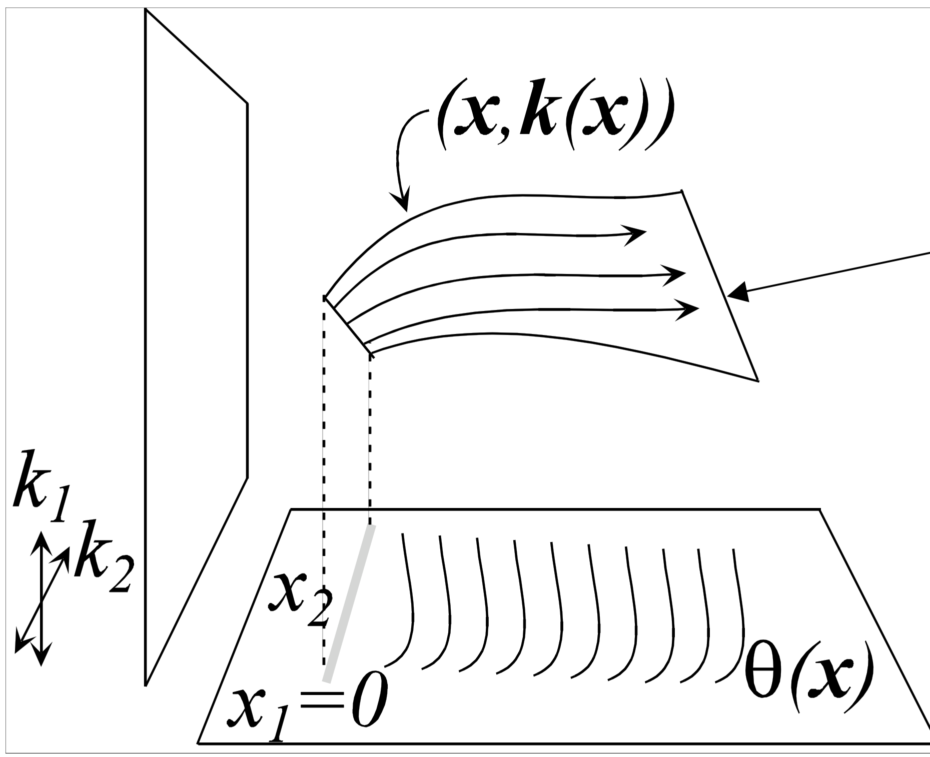
$$\frac{d\mathbf{x}}{d\sigma} = -\frac{\partial D(\mathbf{x},\mathbf{k})}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{d\sigma} = \frac{\partial D(\mathbf{x},\mathbf{k})}{\partial \mathbf{x}}.$$

Define the phase along any ray launched from $(\mathbf{x}_0,\mathbf{k}_0)$ on the boundary:

$$\theta(\sigma) \equiv \theta_0 + \int_0^{\sigma} \mathbf{k}(\sigma') \cdot \frac{d\mathbf{x}(\sigma')}{d\sigma'} d\sigma'$$

Gives $\theta(\mathbf{x})$ for a *family* of rays. The natural place to view this is in ray phase space.

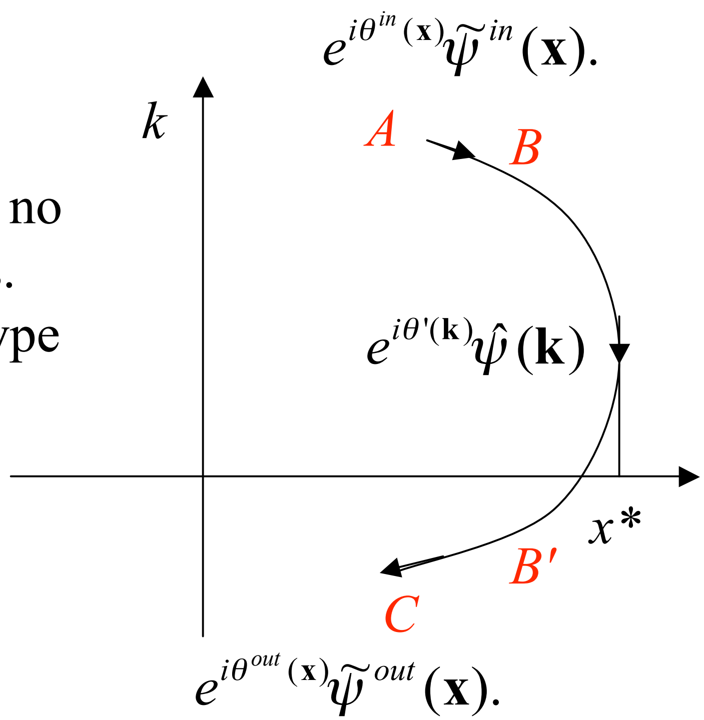
Geometrical pictures and asymptotics



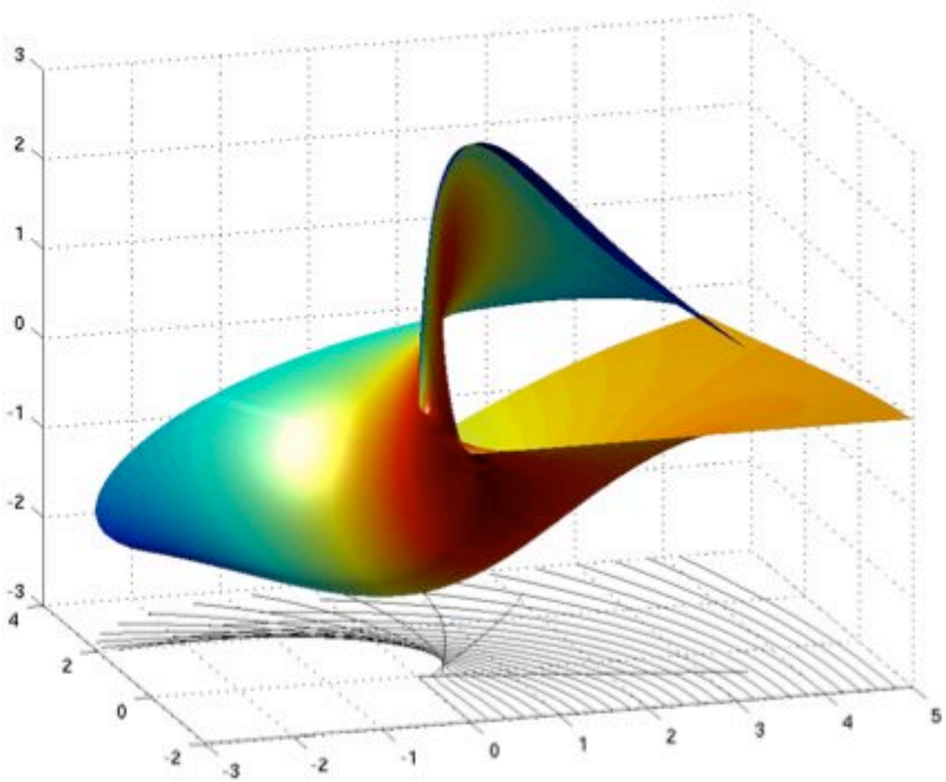
"Lagrange manifold"

Trouble in paradise: caustics
Maslov theory (Keller, Heller, Berry, Kaufman, Friedland, Littlejohn, Delos, Weinstein, and many others…)

- In 1-dimension, caustics are just 'turning points'.
- (A) Incoming wave: use WKB in x-space.
- (B) Change from x- to k-space near x*. There is no singluarity in the k-representation, so WKB works.
- (B') Transform back to x-space: gives an Airy-type function
- (C) Continue using WKB in x-space
- Maslov argued this could be done in arbitrary dimensions.



Multidimensional caustics are far more interesting!



Scattering from a weakly attractive potential with a hard core (Fig. by A. Richardson, based on a paper by Delos)

See Littlejohn, Delos, and others for more on caustics.

Back to amplitude transport (away from caustics)…

$$\left[\underbrace{\nabla_k \mathbf{D} \cdot \nabla}_{symmetrize!}\right] \cdot \left[E(\mathbf{x})\hat{\mathbf{e}}(\mathbf{x})\right] = 0.$$

$\Downarrow$ A little algebra

$$\nabla \cdot \left(J\mathbf{v}\right) = 0, \quad J = D_\omega \left|E\right|^2, \quad \mathbf{v} = -\left.\frac{\nabla_k D}{D_\omega}\right|_{\mathbf{x},\mathbf{k}=\nabla\theta}$$

Wave action flux conservation

The importance of action principles and symmetries (Dewar, Holms, Kaufman, Morrison etc.)

$$A[\psi] = \int d^n\mathbf{x}\, \psi^*(\mathbf{x},t) D(\mathbf{x}, -i\vec{\nabla}, i\vec{\partial}_t)\psi(\mathbf{x},t).$$

$$A'[\tilde{\psi},\theta] = \int d^n\mathbf{x}\, D(\mathbf{x}, \nabla\theta, -\theta_t)|\tilde{\psi}(\mathbf{x},t)|^2. \qquad \psi = e^{i\theta(\mathbf{x},t)}\tilde{\psi}(\mathbf{x},t)$$

$$\frac{\delta A'}{\delta\tilde{\psi}} = 0 \quad \Rightarrow \quad D(\mathbf{x}, \nabla\theta, -\theta_t) = 0.$$

$$\theta(\mathbf{x},t) \rightarrow \theta(\mathbf{x},t) + \alpha$$

$$\frac{\delta A'}{\delta\theta} = 0 \Rightarrow$$ Action flux conservation is due to a Noether symmetry!

Further developments: (McDonald/Kaufman)

- Waves in cavities (early 'quantum chaos')
- Covariant derivation of the wave-kinetic equation for vector fields

$$\int d^n x' \, \mathbf{D}(x,x') \cdot \mathbf{E}(x') = \mathbf{j}(x). \qquad x = (\mathbf{x}, t),\, k = (\mathbf{k}, \omega),$$

$$k \cdot x = \mathbf{k} \cdot \mathbf{x} - \omega t$$

$$\Downarrow$$

$$\left(\int d^n x' \, \mathbf{D}(x,x') \cdot \mathbf{E}(x') \right) \tilde{\mathbf{E}}^*(x'') = \mathbf{j}(x) \tilde{\mathbf{E}}^*(x'').$$

Covariant derivation of the wave-kinetic equation (McDonald/Kaufman)

$$\hat{\mathbf{D}}\left|\mathbf{E}\right\rangle\left\langle\mathbf{E}\right| = \left|\mathbf{j}\right\rangle\left\langle\mathbf{E}\right|.$$

Abstract version

Weyl symbol calculus

$$\int d^n x' \langle$$

$$= \int d^n x' \, \mathbf{D}(x, x \cdot \mathbf{E}(x')\widetilde{\mathbf{E}}^*(x'') = \mathbf{j}(x)\widetilde{\mathbf{E}}^*(x'').$$

$$\mathbf{D}(x,k) * \mathbf{W}(x,k) = \mathbf{S}_{jE}(x,k).$$

Covariant derivation of the wave-kinetic equation (McDonald/Kaufman)

$$\mathbf{D}(x,k) * \mathbf{W}(x,k) = \underbrace{\mathbf{D}(x,k)\mathbf{W}(x,k)}_{O(0)} - \frac{i}{2}\underbrace{\{\mathbf{D},\mathbf{W}\}}_{O(1)} + \ldots = \underbrace{\mathbf{S}_{jE}(x,k)}_{O(1)}.$$

$$O(0): \quad \mathbf{D}(x,k)\mathbf{W}(x,k) = 0.$$

Can simultaneously diagonalize using e-vectors of **D**.

$$O(0): \quad D_\alpha(x,k)W_\alpha(x,k) = 0 \quad \Rightarrow \quad D_\alpha(x,k) = 0 \text{ } or \text{ } W_\alpha(x,k) = 0$$

$$O(1): \quad \{D_\alpha, W_\alpha\} = \frac{dW_\alpha}{d\sigma_\alpha} = S_\alpha(x,k).$$

Wave-kinetic equation (almost)

Mode conversion

The previous WKB derivation assumes that there is only one null eigenvalue (this is hiding in the ordering assumptions).

What happens when that assumption breaks down? Mode conversion!

What is mode conversion?

Plasmas, fluids and other media can support a wide variety of *linear* wave types, with different dispersion characteristics and polarizations.

- In a weakly non-uniform medium, the dispersion characteristics and polarizations are *local* objects.
- For fixed frequency, ω, near a point $\mathbf{x}_*$, wave types '*a*' and '*b*' (with different group velocities and polarizations) can have *nearly equal* wavenumbers.
- Mode conversion is due to a *local phase resonance*.

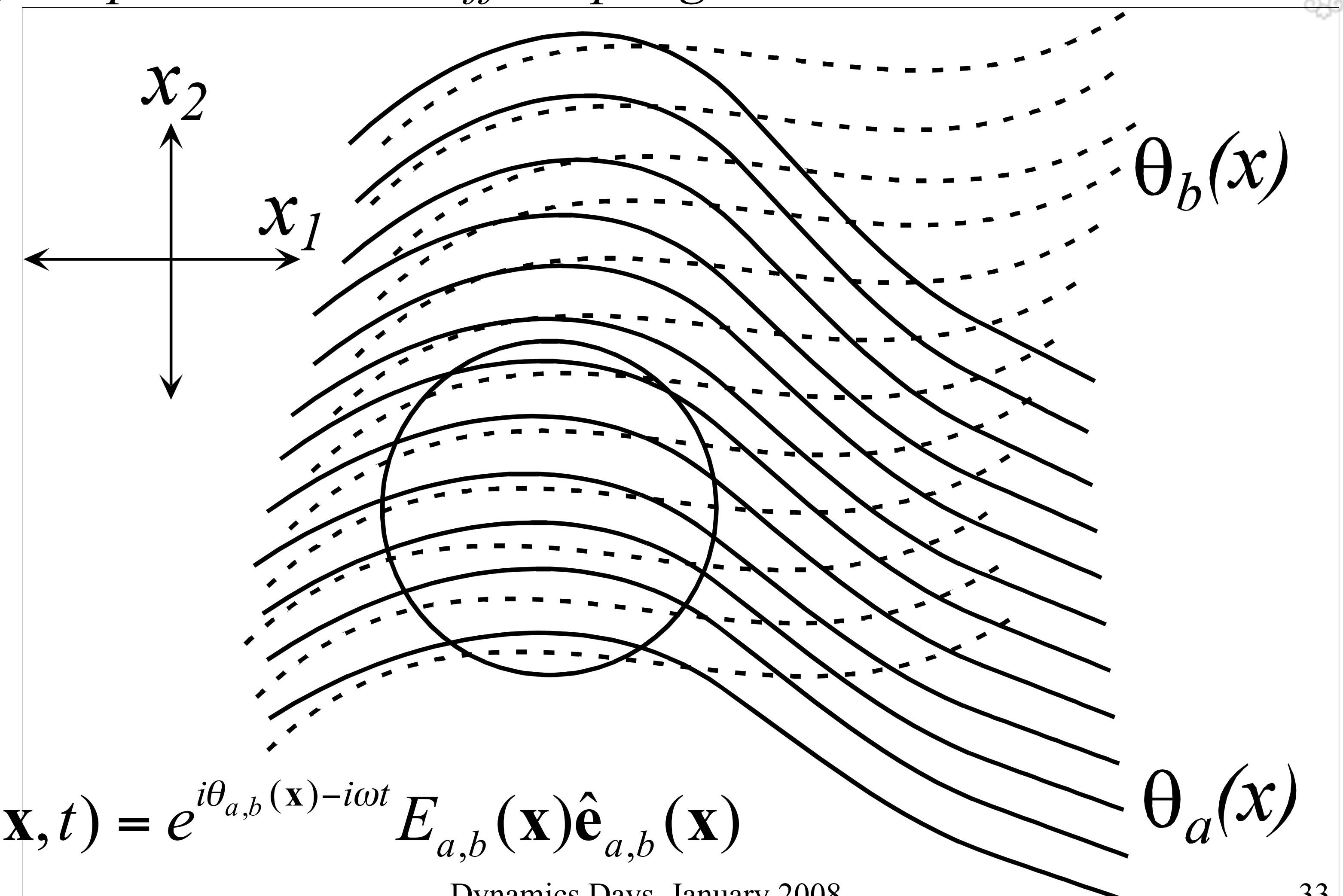
Distinguish two cases:

CASE I) The two waves undergoing conversion have *different* polarizations. This can be reduced locally to a 2-component *vector wave* problem.

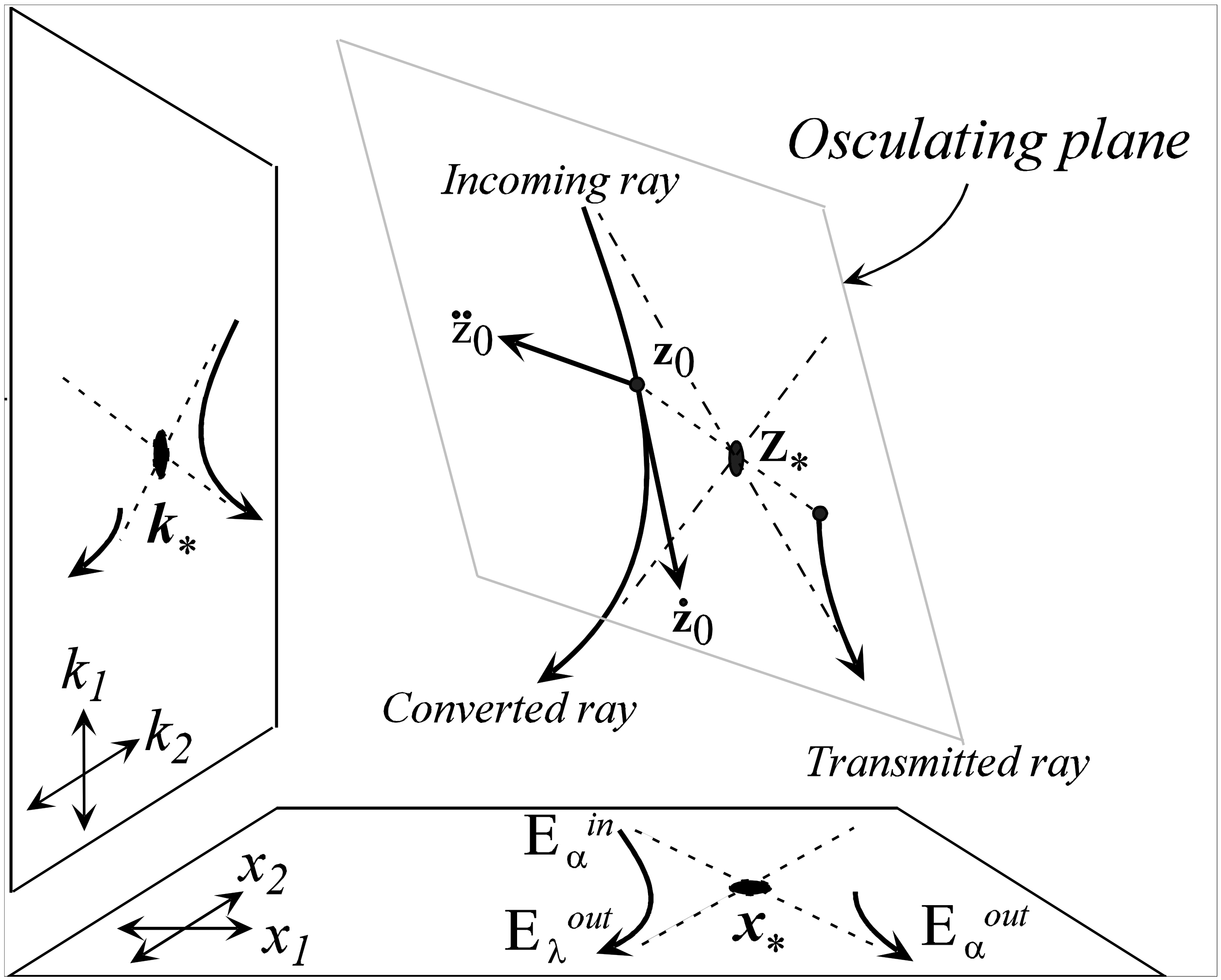
CASE II) The two waves undergoing conversion have *the same* polarization. This can be reduced locally to a *scalar* problem. ("*Landau-Zener*", "*avoided crossings*".)

Our work focuses on CASE I.

Thought experiment: turn off coupling between the two modes



$$\mathbf{E}_{a,b}(\mathbf{x},t) = e^{i\theta_{a,b}(\mathbf{x})-i\omega t} E_{a,b}(\mathbf{x})\hat{\mathbf{e}}_{a,b}(\mathbf{x})$$



Osculating plane
Incoming ray
$\ddot{z}_0$
z_0
z_*
$\dot{z}_0$
Converted ray
Transmitted ray
k_*
k_1
k_2
E_α^{in}
E_λ^{out}
x_*
E_α^{out}
x_2
x_1

Linear mode conversion occurs in many areas of physics

- RF heating in plasmas
- Ionospheric physics
- Atomic, molecular and nuclear physics

(Landau-Zener crossings, spin-orbit resonance)

- Geophysics *(e.g.* equatorial waves*)*
- Neutrino physics ('MSW effect')
- Black hole theory
- Solid mechanics
- Magnetohelioseismology

Friedland and Kaufman (PRL, Phys. Fl. 1987)

$$\int d\mathbf{x}' dt' \, \mathbf{D}(\mathbf{x}, \mathbf{x}', t - t') \cdot \mathbf{E}(\mathbf{x}', t') = 0.$$

$$\Downarrow$$

$$\mathbf{D}(\mathbf{x}, -i\nabla, i\partial_t) \cdot \mathbf{E}(\mathbf{x}, t) = 0$$

Congruent reduction procedure identifies uncoupled polarizations: $\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b$

$$\mathbf{E}(\mathbf{x}, t) = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \left[\hat{\mathbf{e}}_a E_a(\mathbf{x}) + \hat{\mathbf{e}}_b E_b(\mathbf{x}) \right]$$ *New ansatz*

Friedland and Kaufman (PRL, Phys Fl. 1987)

$$\begin{pmatrix} \hat{D}_{aa} & \hat{D}_{ab} \\ \hat{D}^{*t}_{ab} & \hat{D}_{bb} \end{pmatrix} \begin{pmatrix} E_a \\ E_b \end{pmatrix} = 0.$$

Galerkin reduction from NXN to 2X2

$$\hat{D}_{aa} \equiv \hat{\mathbf{e}}^{*t}_a \cdot \mathbf{D}(\mathbf{x}, \mathbf{k} - i\nabla, \omega) \cdot \hat{\mathbf{e}}_a .$$

$$\hat{D}_{ab} \equiv \hat{\mathbf{e}}^{*t}_a \cdot \mathbf{D}(\mathbf{x}, \mathbf{k} - i\nabla, \omega) \cdot \hat{\mathbf{e}}_b = \hat{D}^{*t}_{ab} .$$

Friedland and Kaufman (PRL, Phys Fl. 1987)

$$\begin{pmatrix} -i\mathbf{V}_a\cdot\nabla_x+(\mathbf{x}-\mathbf{x}_*)\cdot\mathbf{R}_a & \eta \\ \eta^* & -i\mathbf{V}_b\cdot\nabla_x+(\mathbf{x}-\mathbf{x}_*)\cdot\mathbf{R}_b \end{pmatrix}\begin{pmatrix} E_a(\mathbf{x}) \\ E_b(\mathbf{x}) \end{pmatrix}=0.$$

Multidimensional

$\mathbf{x}_*$ is a point on the 'conversion surface'..

$$\frac{|E_a^{out}|}{|E_a^{in}|}=\exp\left(-\pi|\eta|^2/B\right)=\tau$$

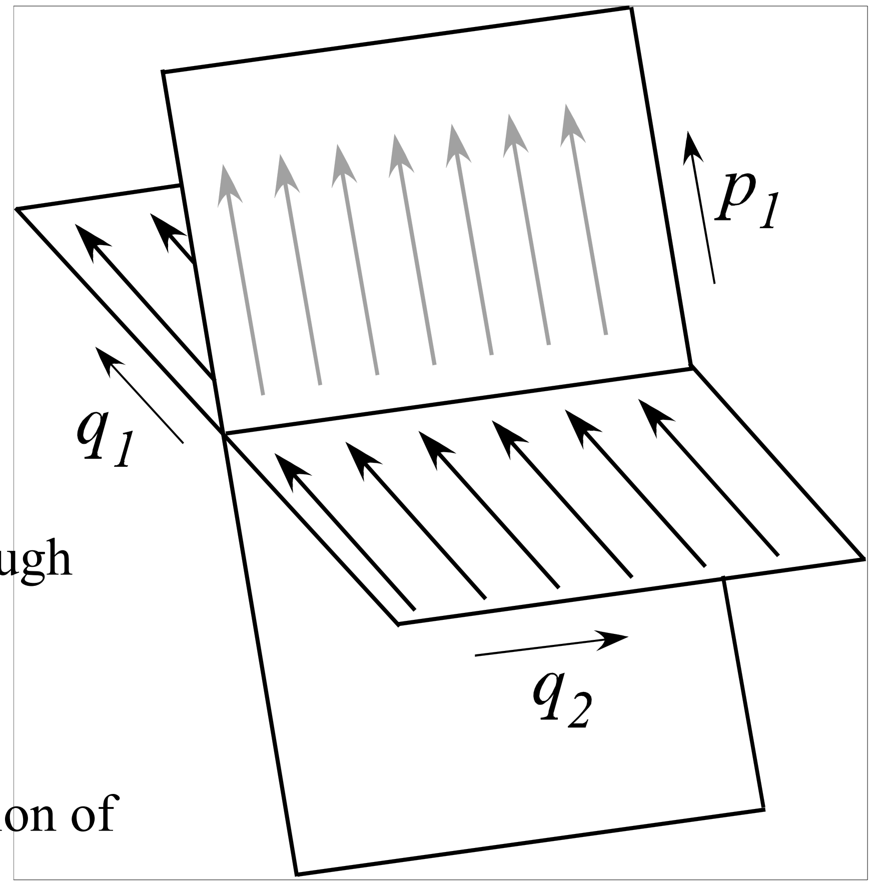
Poisson bracket.!!!

$$B=\left|\mathbf{V}_a\cdot\mathbf{R}_b-\mathbf{V}_b\cdot\mathbf{R}_a\right|=\left|\left\{D_{aa},D_{bb}\right\}\right|$$

Basic insight: Poisson brackets = phase space

$$B = \left|\left\{D_{aa}, D_{bb}\right\}\right|$$

$$= \left|\frac{dD_{aa}}{d\sigma_b}\right| = \left|\frac{dD_{bb}}{d\sigma_a}\right|$$



Conversion occurs 'ray by ray', even though WKB is not valid in conversion regions!

Tracy and Kaufman, PRE (1991)
Multidimensional conversion, computation of Wigner functions

Solution

Using a generalization of the fourier transform (metaplectic transforms), we can replace

$$\begin{pmatrix} -i\mathbf{V}_a \cdot \nabla_x + (\mathbf{x} - \mathbf{x}_*)\cdot \mathbf{R}_a & \eta \\ \eta^* & -i\mathbf{V}_b \cdot \nabla_x + (\mathbf{x} - \mathbf{x}_*)\cdot \mathbf{R}_b \end{pmatrix} \begin{pmatrix} E_a(\mathbf{x}) \\ E_b(\mathbf{x}) \end{pmatrix} = 0.$$

(coupled PDEs with non-constant coefficients), with

$$\begin{pmatrix} i\partial_{q_1} & \eta \\ \eta^* & q_1 \end{pmatrix} \begin{pmatrix} \psi_a(q_1;q_2) \\ \psi_b(q_1;q_2) \end{pmatrix} = 0.$$

a 1st order ODE!

Friedland and Kaufman, Phys Lett A

Tracy and Kaufman, PRE

Solution gives the WKB connection coefficients (Tracy, Kaufman, PRE 1993):

$$\tau(\eta) \equiv \exp(-2\pi\,|\,\eta\,|^2), \quad \beta(\eta) = \frac{(2\pi\tau)^{1/2}}{\eta\Gamma(-i\,|\,\eta\,|^2)}.$$

‘transmission’ ‘conversion’

These are applied *ray-by-ray*.

Applications: for a practical algorithm

- Using ray-based methods for conversion requires:
  - 1. Detection/verification of conversion
  - 2. Finding the transmitted rays and local uncoupled polarizations
  - 3. Galerkin reduction to local 2X2 form and the local coupling const.
  - 4. Local field structure and fitting to incoming/outgoing fields

Sample applications:

- RF heating in fusion devices: ray tracing algorithms (Jaun, et al.)
- Kinetic effects (wave absorption and emission by particles) (Cook, et al.)
- Upwelling in the Gulf of Guinea (Morehead, et al.)
- Neutrino physics (MSW effect) (Brizard, et al.)
- Magnetohelioseismology (Cally, et al.)

Ray tracing for ICRF, including mode conversion and caustics

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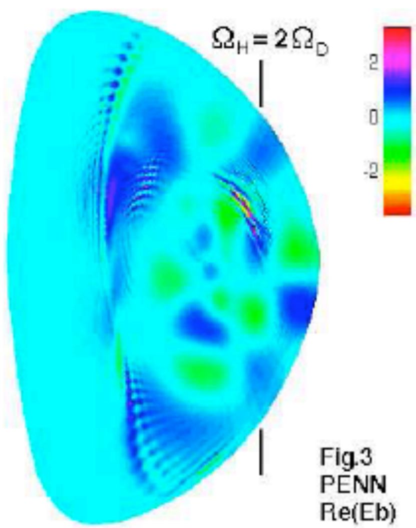
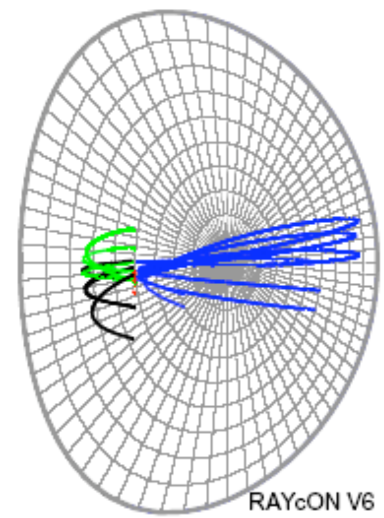
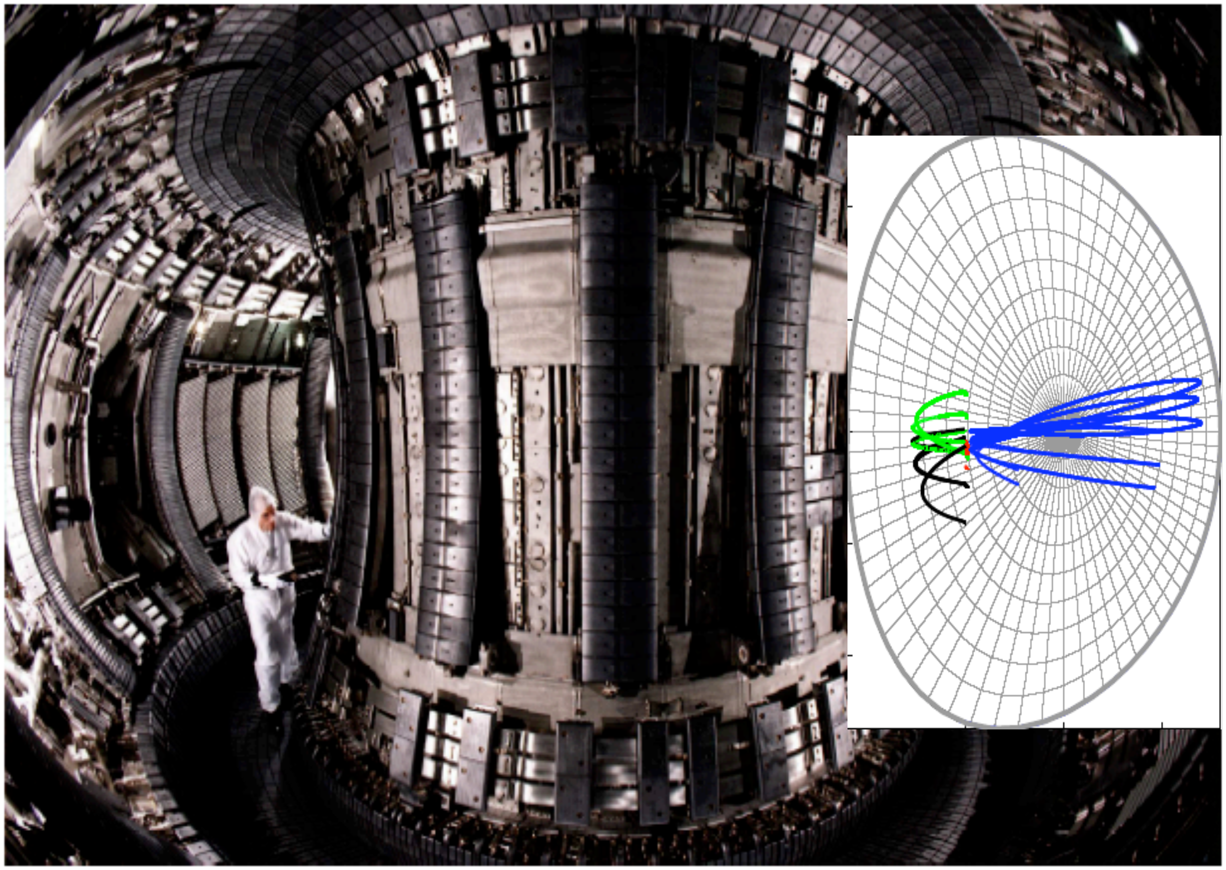


$\Rightarrow$ $\times$1000 speedup $\Rightarrow$



Heating with waves in the Joint European Torus (JET, Oxford).



Electro-magnetic power

- cyclotron resonance
  ions $\approx$ 50 MHz
  electrons $\approx$ 100 GHz
- 32 MW power injected

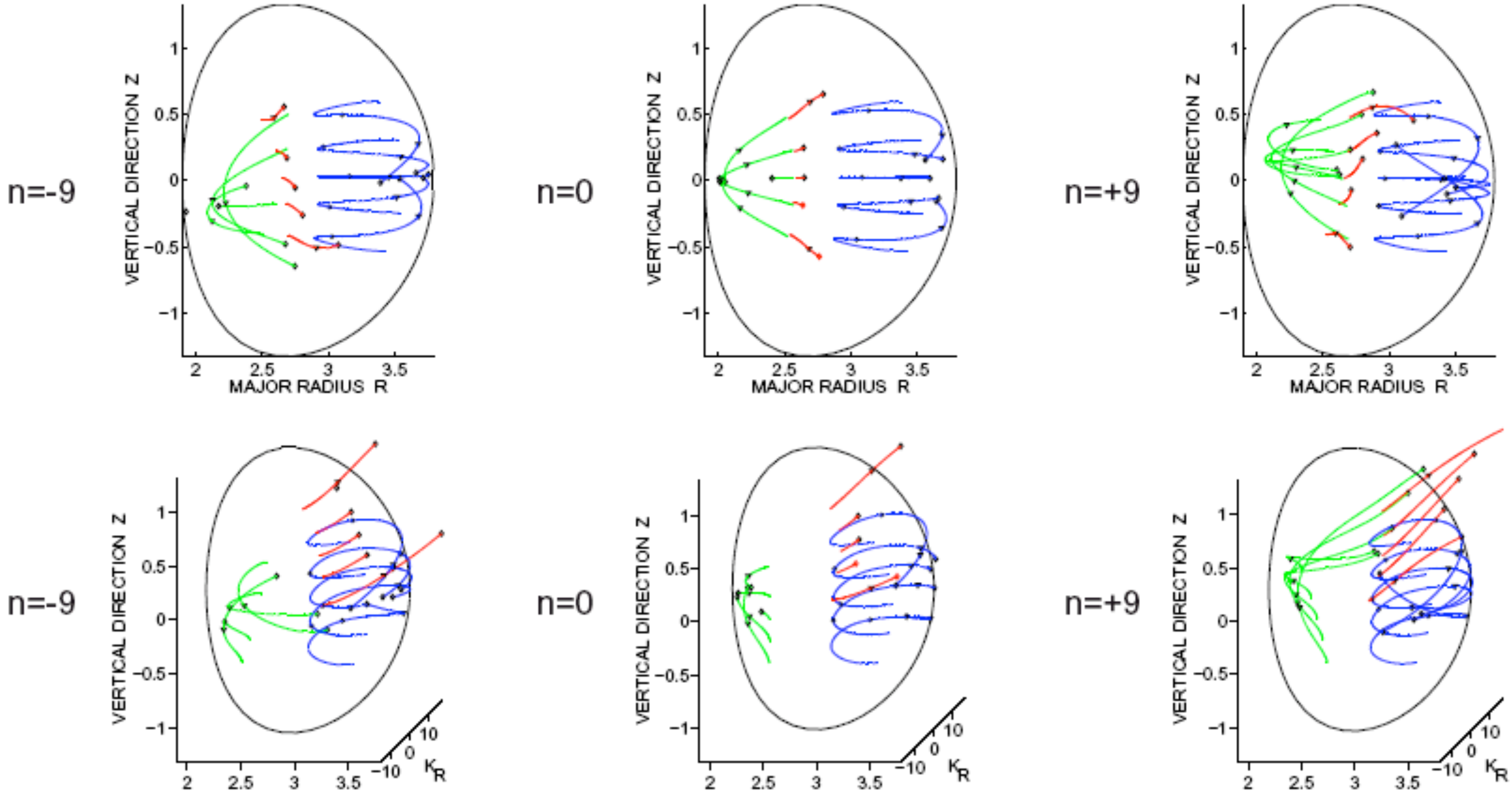
Where is it deposited?

- core / edge?
- electrons / ions?
- heat / current?

$\Rightarrow$ need modelling

- full-wave
- ray-tracing

Result – mode conversion in a JET-like D(55%)-H(45%) plasma.



This is a ‘leaky’ dynamical system, but the leak is in *k*-space, not *x*-space.

Why use ray tracing when there are ‘full-wave’ codes?

- Ray tracing provides valuable physical insight. For example, easier to see where the energy is flowing and how small-scale structures develop *(e.g.* caustics*)*.
- Ray tracing should permit a wider range of parameter studies than full-wave methods (ODEs vs. PDEs).
- Complements and helps verify full-wave results.

Achieved so far:

- First ray-tracing calculations featuring mode conversion in a tokamak
- The algorithm evolves trajectory, amplitude and phase simultaneously
- Ray tracing achieves in seconds what takes hours for global codes
- The results are in qualitative agreement with measurements from JET

Future work:

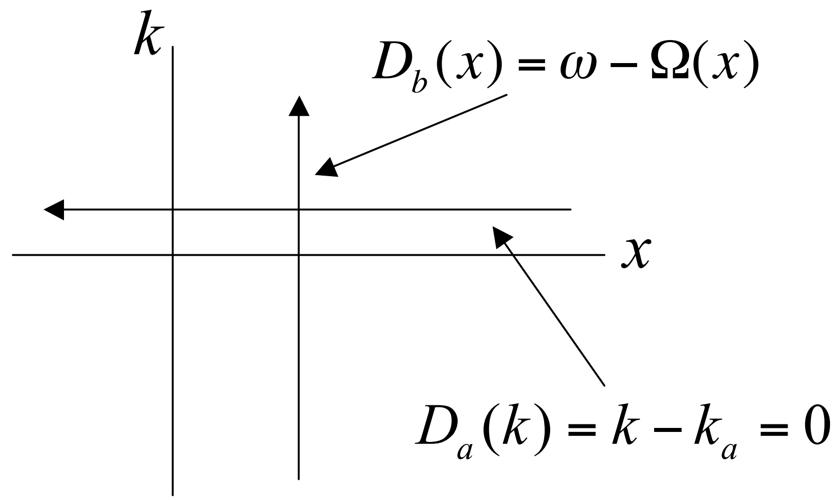
- Compare with full-wave codes and experimental data
- Implement as a heating module in a comprehensive tokamak simulation code
- Warm plasma model, mode conversion to ion / electron Bernstein waves
- Existence of localized wavefield structures predicted by PENN?

Resonance crossing, and emission for minority ions in a non-uniform magnetized plasma

Friedland/Kaufman: resonance crossing is a form of mode conversion from a collective wave to a *gyroballistic* wave

$$\mathbf{D}(x,-i\partial_x,i\partial_t)\cdot\mathbf{E}(x,t)=\begin{pmatrix}\partial_t+v\partial_x & \eta\\ \eta & \partial_t+i\Omega(x)\end{pmatrix}\begin{pmatrix}E_a(x,t)\\ E_b(x,t)\end{pmatrix}=0$$

$$\mathbf{D}(x,-i\partial_x,\omega)\cdot\mathbf{E}(x)=\begin{pmatrix}-i\partial_x-k_a & \eta\\ \eta & \omega-\Omega(x)\end{pmatrix}\begin{pmatrix}E_a(x)\\ E_b(x)\end{pmatrix}=0,\quad \mathbf{E}(x,t)=e^{-i\omega t}\mathbf{E}(x)$$

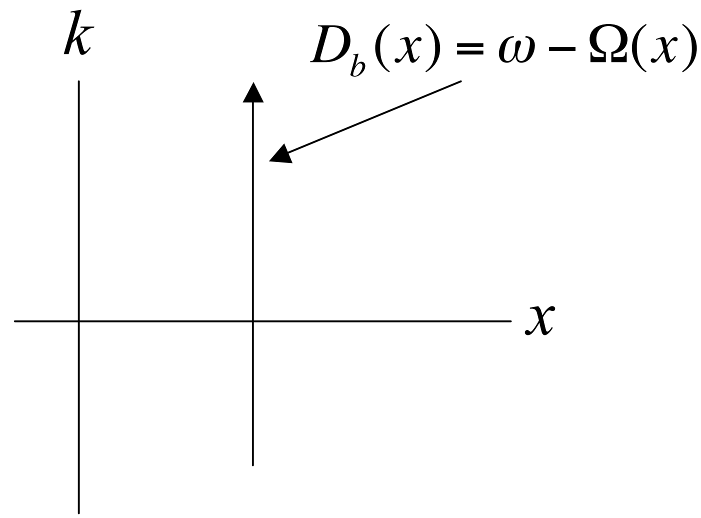


$$\mathbf{D}(x,k,\omega) = \begin{pmatrix} k - k_a & \eta \\ \eta & \omega - \Omega(x) \end{pmatrix}.$$

Gyroballistic waves propagate in *k*-space

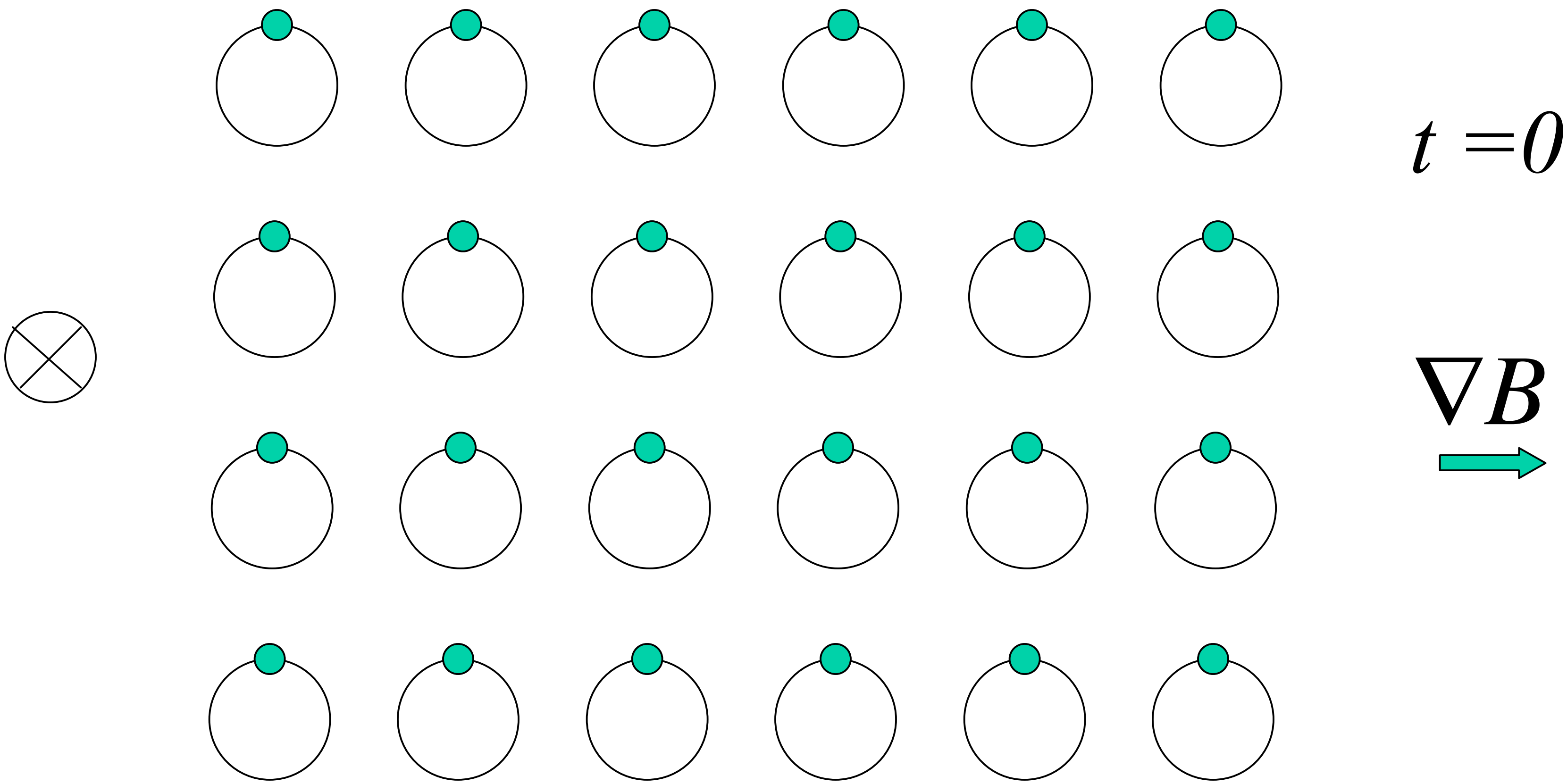
$$\dot{x} = -\frac{\partial D_b}{\partial k} = 0,$$

$$\dot{k} = \frac{\partial D_b}{\partial x} = -\Omega'(x)$$

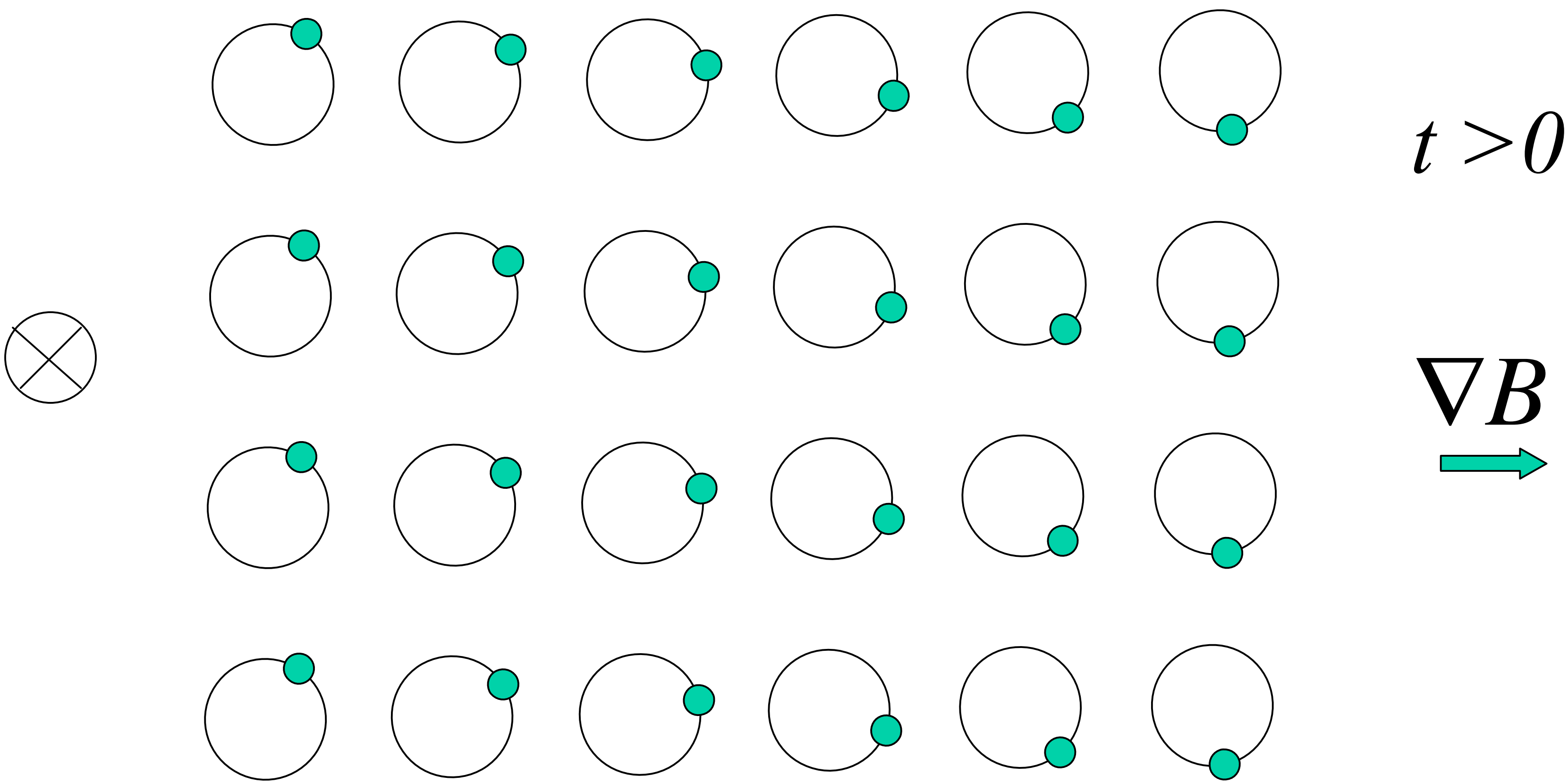


$$\mathbf{D}(x, -i\partial) \cdot \mathbf{E}(x) = \begin{pmatrix} -i\partial - k_a & \eta \\ \eta & \omega - \Omega(x) \end{pmatrix} \begin{pmatrix} E_a(x) \\ E_b(x) \end{pmatrix} = 0$$

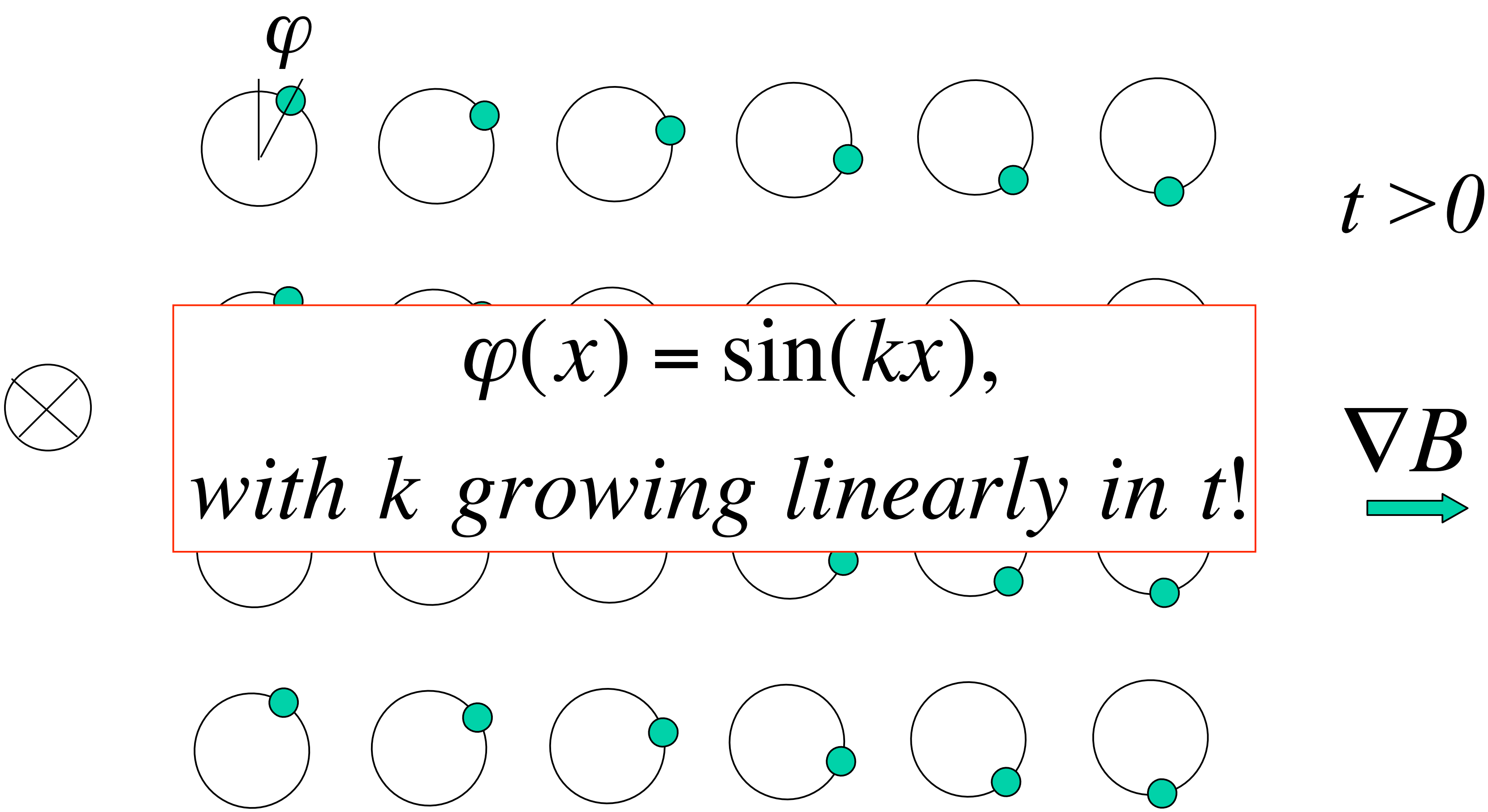
Gyroballistic waves: what are they?



Gyroballistic waves: what are they?



Gyroballistic waves: what are they?

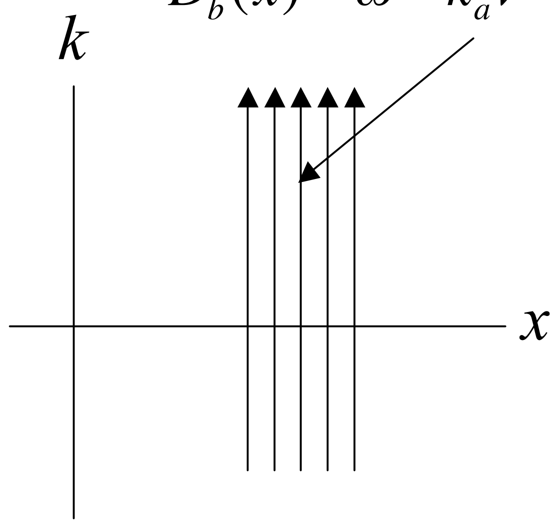


$$\varphi(x) = \sin(kx),$$

with k growing linearly in t!

Doppler effects due to ion motion along B means we get a continuum of resonances!

$$D_b(x) = \omega - k_a v - \Omega(x) = 0$$

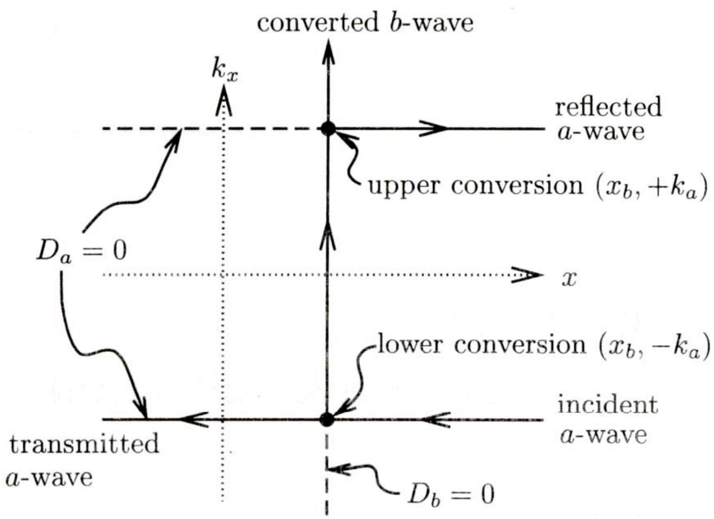


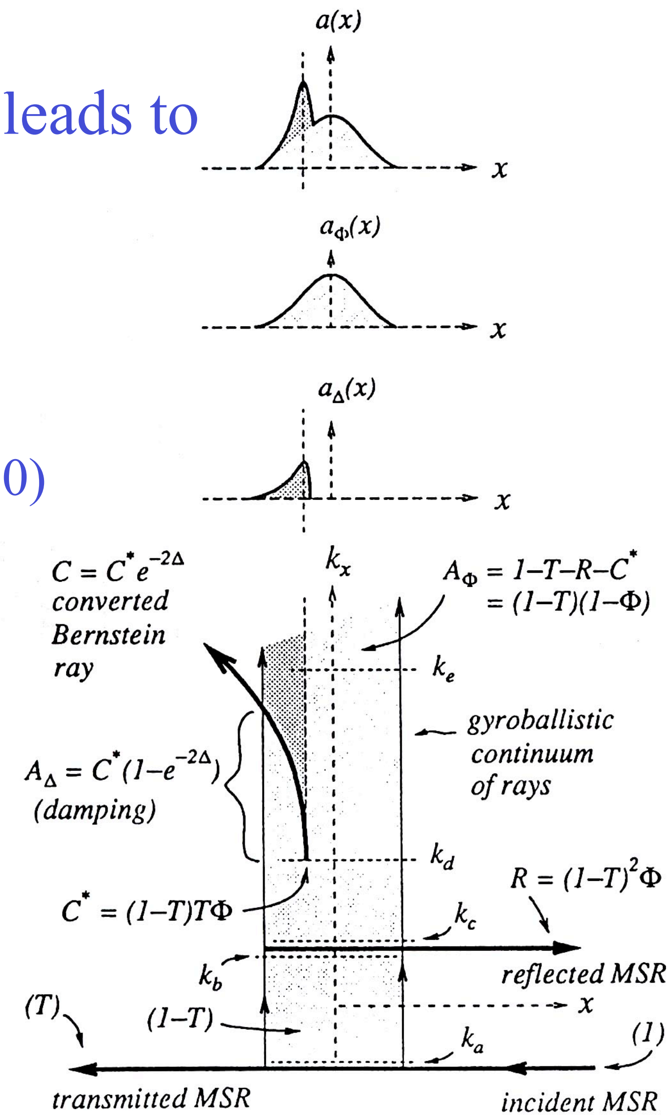
Cooke et al. PHYSICS LETTERS A *(1993)*

1] Interactions via the electric field of E_a leads to phase synchronization!

2] The synchronization pattern is a wave (like Smokey Mountain Fireflies?)

(see, e.g. Strogatz's work using Vlasov models, "From Kuramoto to Crawford…", Physica D 2000)





J. Plasma Physics (1996), vol. 55, part 3, pp. 449–486 (1996)

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Generalized Case–van Kampen modes in a multidimensional non-uniform plasma with application to gyroresonance heating

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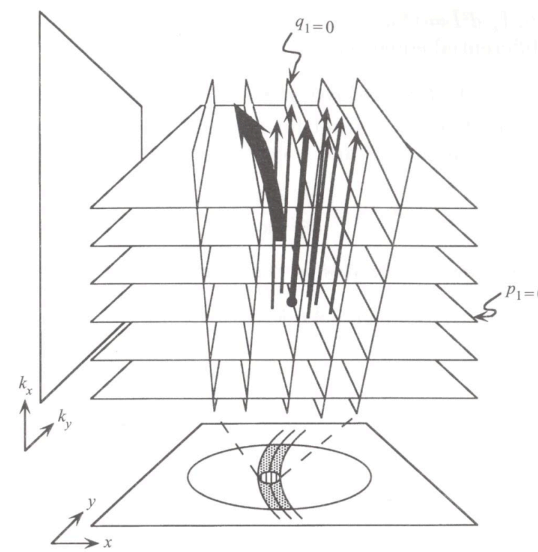


FIGURE 8. Schematic of the nature of the free gyroballistic and Bernstein rays in the interior region. The reference point **X** lies within the resonance layer. The asymptotic expansion examines the self-consistent dynamics within the resonance zone directly above this region. The 'guide ray' is the bold black line that starts at the intersection of the planes $p_1 = 0$ and $q_1 = 0$. As it propagates, it continues to lie within the surface $q_1 = 0$. (The surfaces of constant p_1 are horizontal; those of constant q_1 are nearly vertical.) Neighbouring rays that both start nearby in phase space and have similar values for **I** propagate nearly parallel to the guide ray. These are represented by the straight grey arrows surrounding the guide ray. The Bernstein wave emerges from this ray bundle due to their self-interactions. This collective wave is indicated by the bold grey arrow.

WKB+
Mode conversion+
Case-van Kampen+
Bateman-Kruskal+
Crawford-Hislop=
Analytic theory for excitation of the Bernstein wave

Multidimensional normal form

Littlejohn and Flynn showed that, in multidimensions, we must keep the linear terms in the off-diagonal, too!

$$\begin{pmatrix} -i\mathbf{V}_a \cdot \nabla_x + \left(\mathbf{x}-\mathbf{x}_*\right)\cdot \mathbf{R}_a & \eta + h.o.t. \\ \eta^* + h.o.t.^* & -i\mathbf{V}_b \cdot \nabla_x + \left(\mathbf{x}-\mathbf{x}_*\right)\cdot \mathbf{R}_b \end{pmatrix} \begin{pmatrix} E_a(\mathbf{x}) \\ E_b(\mathbf{x}) \end{pmatrix} = 0.$$

See also Colin de Verdiere; Braam and Duistermaat; Emmrich and Weinstein

We can perform a combination of congruence and canonical transformations to find the normal form (Tracy and Kaufman, PRL):

$$\mathbf{D}(\mathbf{z}') = \begin{pmatrix} q_1 & \eta + q_2 + i\kappa p_2 \\ \eta^* + q_2 - i\kappa p_2 & p_1 \end{pmatrix} + h.o.t. \qquad \kappa = \frac{\omega}{\gamma}$$

Notice that the diagonals Poisson-commute with the off-diagonals. Conjecture: we can take this as the *definition* of the normal form and extend to higher order.

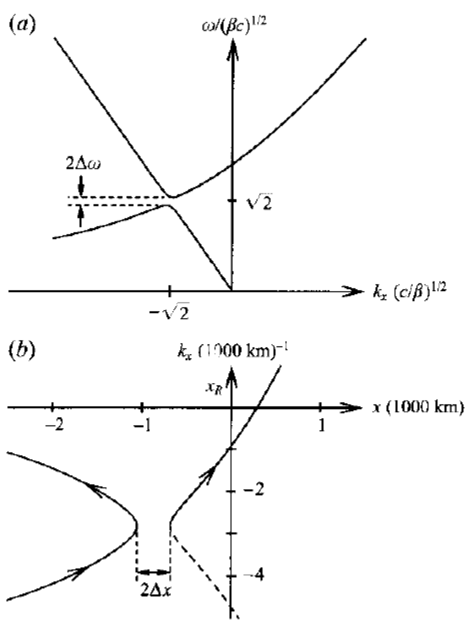
The related 2X2 wave equation

$$\begin{pmatrix} \hat{q}_1 & \eta + \hat{q}_2 + i\kappa\hat{p}_2 \\ \eta^* + \hat{q}_2 - i\kappa\hat{p}_2 & \hat{p}_1 \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = 0$$

can be solved by separation of variables and a generalization of the Fourier transformation.

Conversion in the Gulf of Guinea (JFM, 1999)

- *Shallow water theory*
- *Non-uniform thermocline*
- *Seasonal upwelling*



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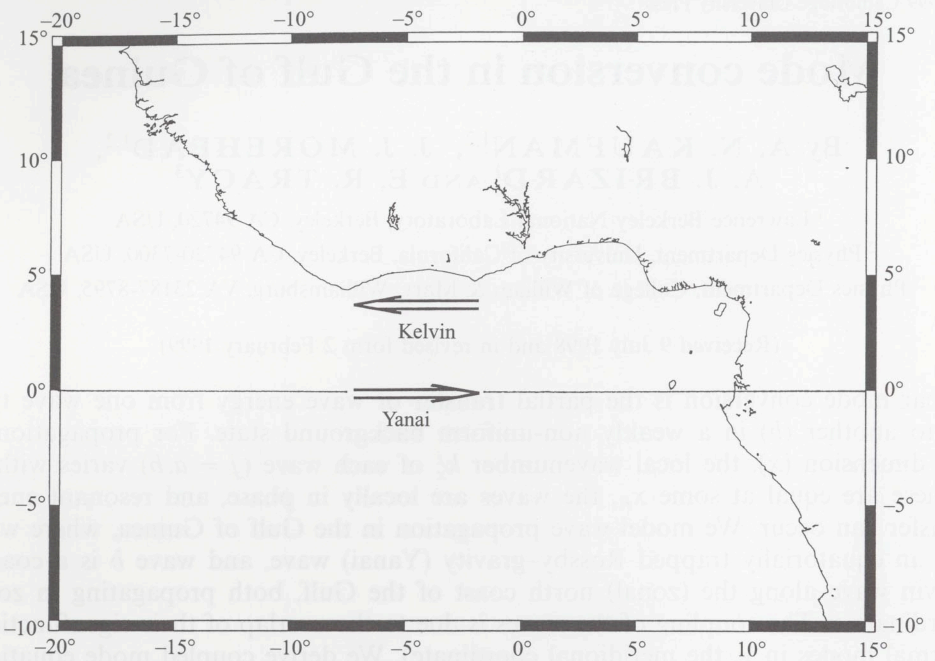
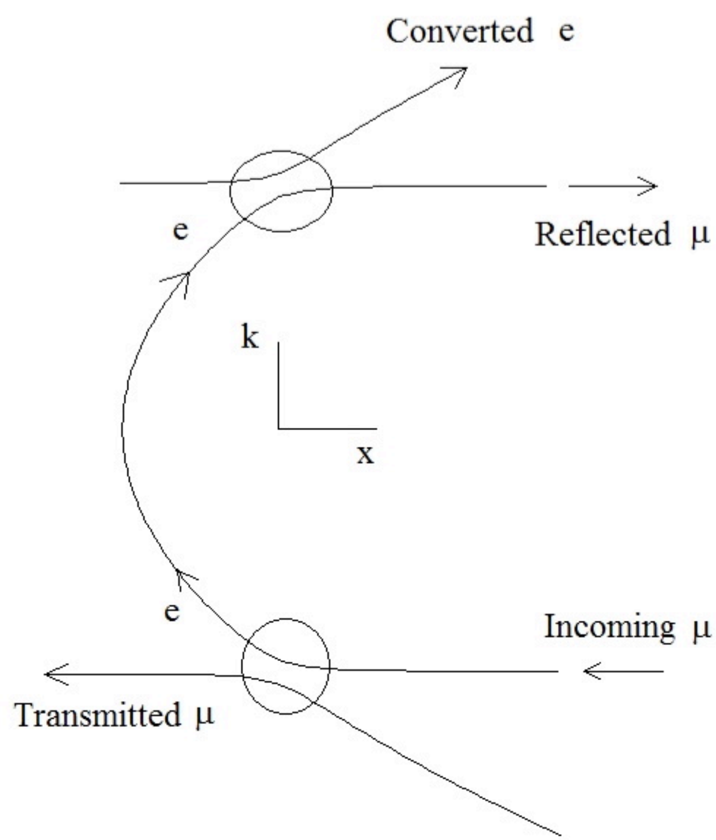


FIGURE 1. The Gulf of Guinea with a Yanai wave propagating eastwards along the equator and a Kelvin wave propagating westwards along the south coast of West Africa.

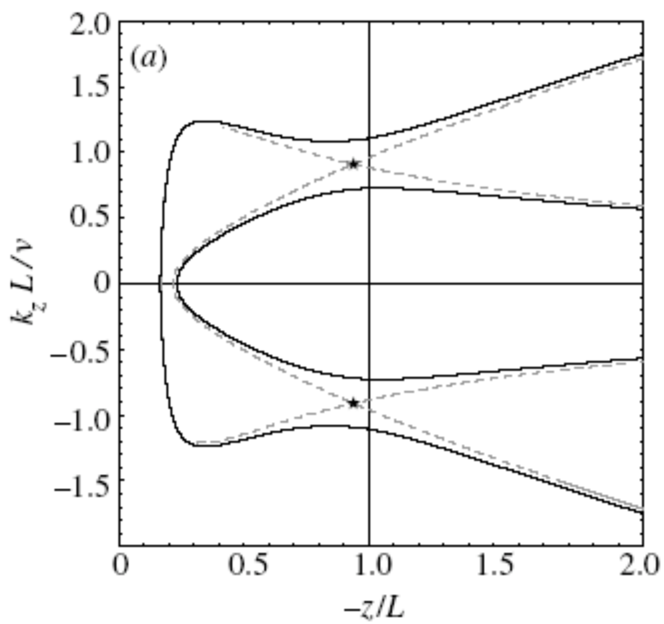
Neutrinos (MSW effect)

R. Bingham, R.A. Cairns, J.M. Dawson, et al., Phys. Lett. A 232 (1997)257.

Brizard, unpublished



Magnetohelioseismology (Cally)



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Dispersion relations, rays and ray splitting in magnetohelioseismology

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Work in progress:

- Effects of higher order terms (Richardson poster)
- Application of the ray-based algorithm to realistic tokamak scenarios (Jaun talk)
- Direct comparisons of full-wave and ray-based approaches to conversion (Xiao, Richardson)
- Path integral methods for non-standard conversions? (Zobin, Richardson PhD thesis, 2007)
- Constructing 2X2 normal form following a ray, rather than extending order by order in Taylor expansion: 'semi-holonomic' dynamics
- Nonlinear effects (auotresonance, Friedland)