The linked-twist map approach to modeling fluid flows having strong mixing properties on domains of full measure

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Why study linked-twist maps?

Linked-twist maps provide some of the few known examples of explicitly defined systems which

- are non-uniformly hyperbolic,
- preserve the Lebesgue measure (area),
- are isomorphic to Bernoulli shifts (strongly mixing).

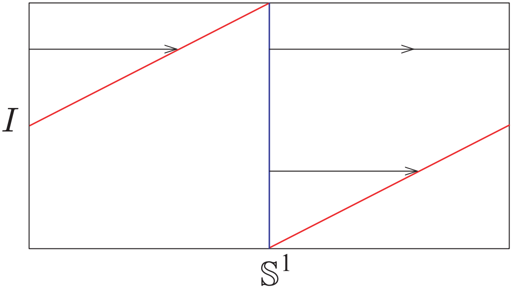
Moreover, they are physically realistic as models of some important fluid flows, examples of which we will see today.

Outline of today's talk

We will begin by describing the elementary building block known as a *twist map*, before looking at

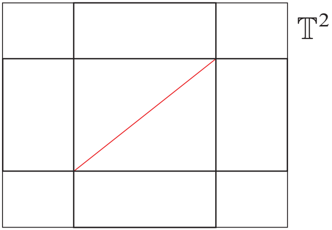
- Linked-twist maps on the two-torus and their applications
- Linked-twist maps in the plane and their applications
- A novel method to overcome problems in analysing the latter

Twist maps



- Homeomorphism of the cylinder $\mathbb{S}^1 \times I$
- Boundaries are invariant
- We illustrate a 'linear' single twist, showing initial conditions (blue) and their images (red)

Simulation of a toral linked-twist map



- Overlap two cylinders to create a manifold contained in the two-torus
- A linked-twist map is the composition of two twist maps
- It is possible to prove the Bernoulli property for such maps, owing to the natural coordinate system with which we define them

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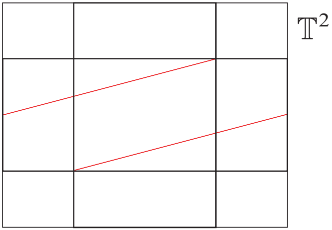

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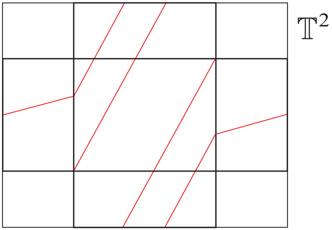

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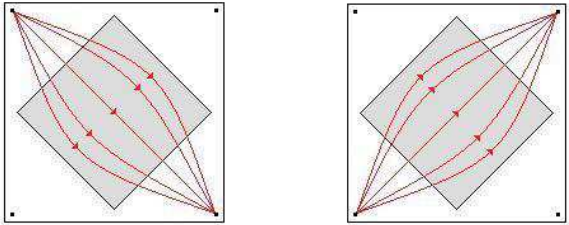
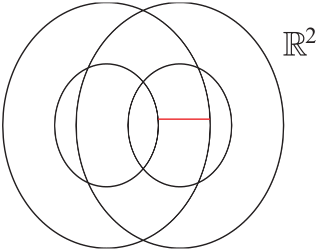
An application: DNA microarrays



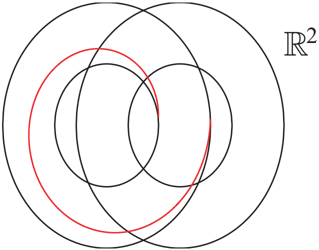
- Shaded region: array of probes, each containing a small DNA molecule composed of a few nucleotide bases
- Labelled DNA injected into the region alternately from top-left and bottom-left, driven by a pressure gradient
- *Hybridization* occurs when labelled DNA combines with its complementary sequence on a probe; the degree of similarity between the species may then be determined

Simulation of a planar linked-twist map



- Manifold resembles two overlapping donuts
- As before, our linked-twist map is the composition of two twist maps
- The combination of coordinate systems required to describe these maps make them more difficult to study than their toral counterparts

Simulation of a planar linked-twist map



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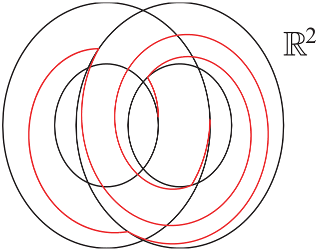

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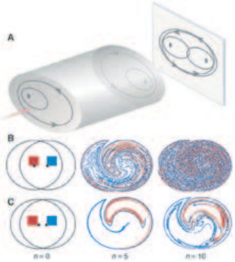
An application: Pipe mixer



A Schematic of channel-type micromixer

B LTM mechanism causes the flow to mix completely after five periodic elements

C Flow exhibits islands, resulting in poor, incomplete mixing

Figure courtesy of Ottino and Wiggins in Science

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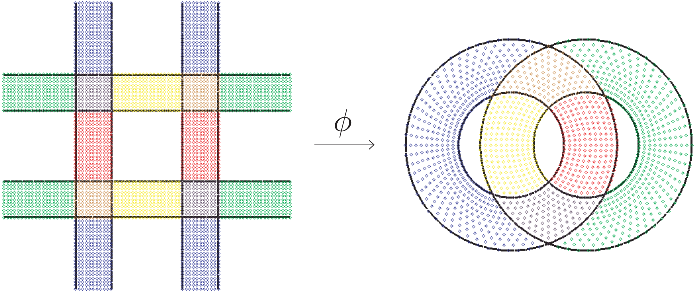
Our contribution: overcoming coordinate difficulties



- Introduce a semi-conjugacy ϕ so that

$$\phi \circ H_{\text{torus}} = H_{\text{plane}} \circ \phi$$

- Prove the Bernoulli property on the torus where we have a global coordinate system
- Ornstein (1971): *Factors of Bernoulli shifts are themselves Bernoulli shifts*

Summary and outlook

- Linked-twist maps model a number of ‘mixing’ devices in which crossing of streamlines is a central ambition
- Mathematically, they are one of a small number of examples of non-uniformly hyperbolic systems for which we can draw conclusions about the mixing properties
- There are many interesting and open questions relating to their dynamics