Renormalization group method for predicting frequency clusters in a chain of nearest-neighbor Kuramoto oscillators.

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Outline:

1. Background and motivation

2. RG steps

3. Numerical RG

4. Results

1 Background

- Big picture - synchronization in nature.

- Well-known model - self-driven phase oscillators with sine coupling and frozen-in disorder of intrinsic frequencies:

$$\dot{\theta}_i = \omega_i + \sum_{j \neq i} K_{ij} \sin\left(\theta_j - \theta_i\right) \qquad i \in \{1, N\}$$

- Special case - nearest-neighbor interaction:

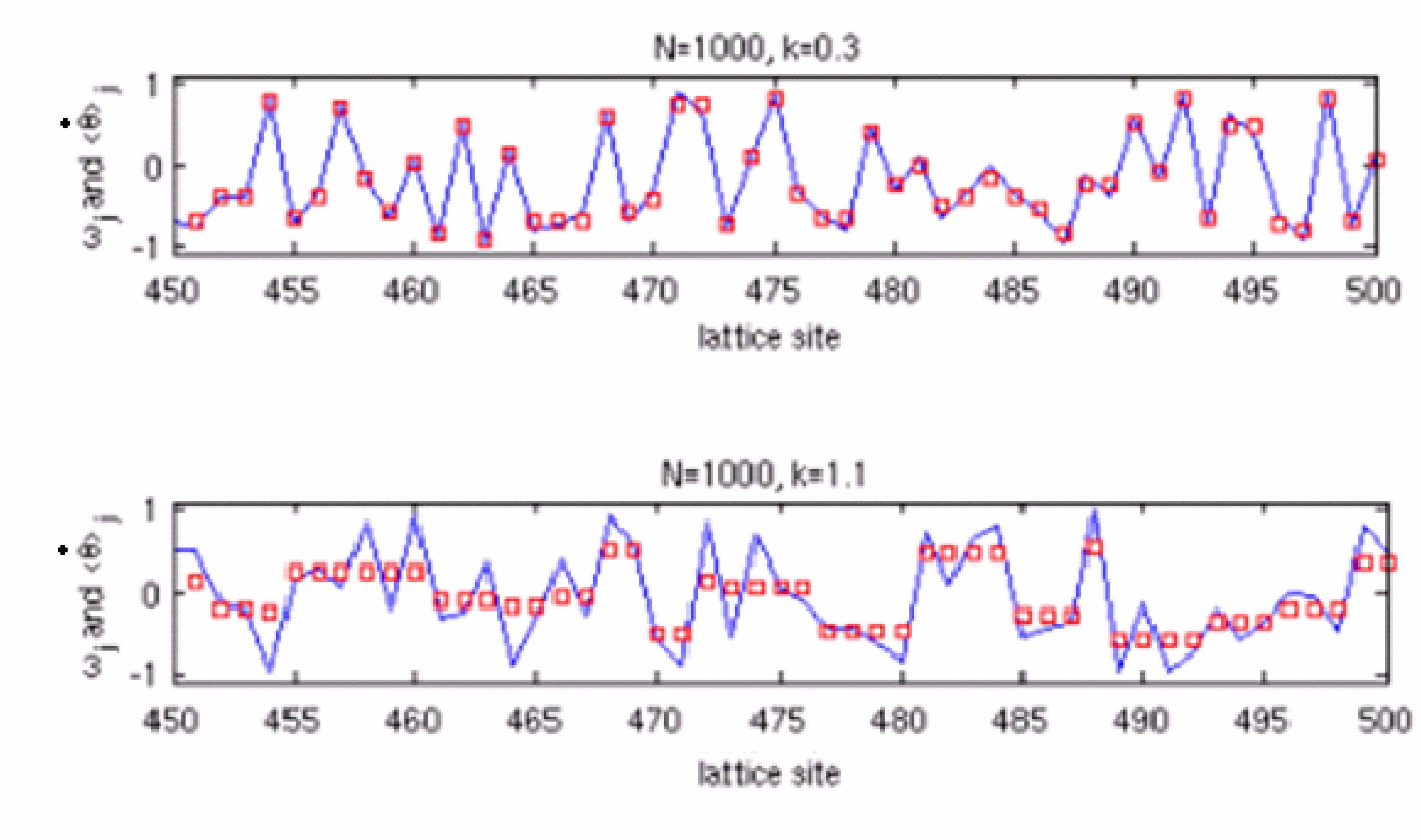
$$\dot{\theta}_i = \omega_i + K \sin\left(\theta_{i-1} - \theta_i\right) + K \sin\left(\theta_{i+1} - \theta_i\right)$$

- Known fact (Strogatz and Mirollo):

$$\lim_{N \to \infty} \left(\text{probability of global synchronization}\right) = 0 \text{ if } \omega_i \text{ are random.}$$

- Another fact: although there is no globally-synchronizing transition, **collective structures** do exist.

- Example: clusters of common frequency, 1-d (Ermentrout and Kopell, 1984).



$$\dot{\theta}_i = \omega_i + K\sin(\theta_{i-1} - \theta_i) + K\sin(\theta_{i+1} - \theta_i)$$

- Frequency clusters is a complicated problem (Strogatz and Mirollo, 1987). Restrict the discussion to 1 dimension.

- Ermentrout and Kopell:

In a chain with a linear frequency profile:

  - proved the existence of limit cycles
  - related them to breaks in frequency clusters
  - predicted sizes of frequency differences between neighboring clusters

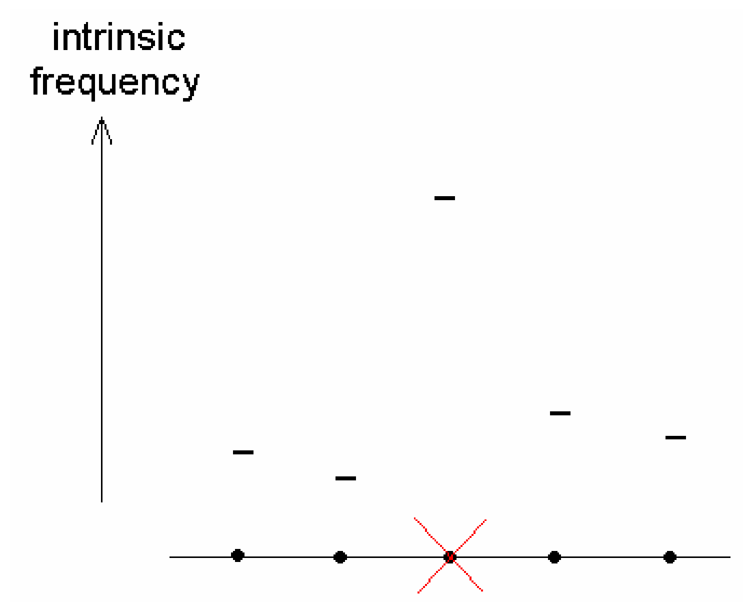
Global, dynamical systems point of view.

- Around the same time, renormalization group method was developed for random 1-d quantum spin chains (Dasgupta and Ma, 1980).

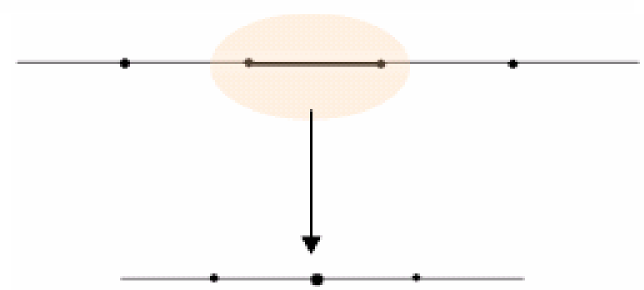
- Recently extended by one of us (E. Altman, Y. Kafri, A. Polkovnikov, **G. Refael**, 2004) to random 1-d Josephson junction arrays.

  This motivated our work.

- Large frequency decimation:



- Strongest coupling decimation:



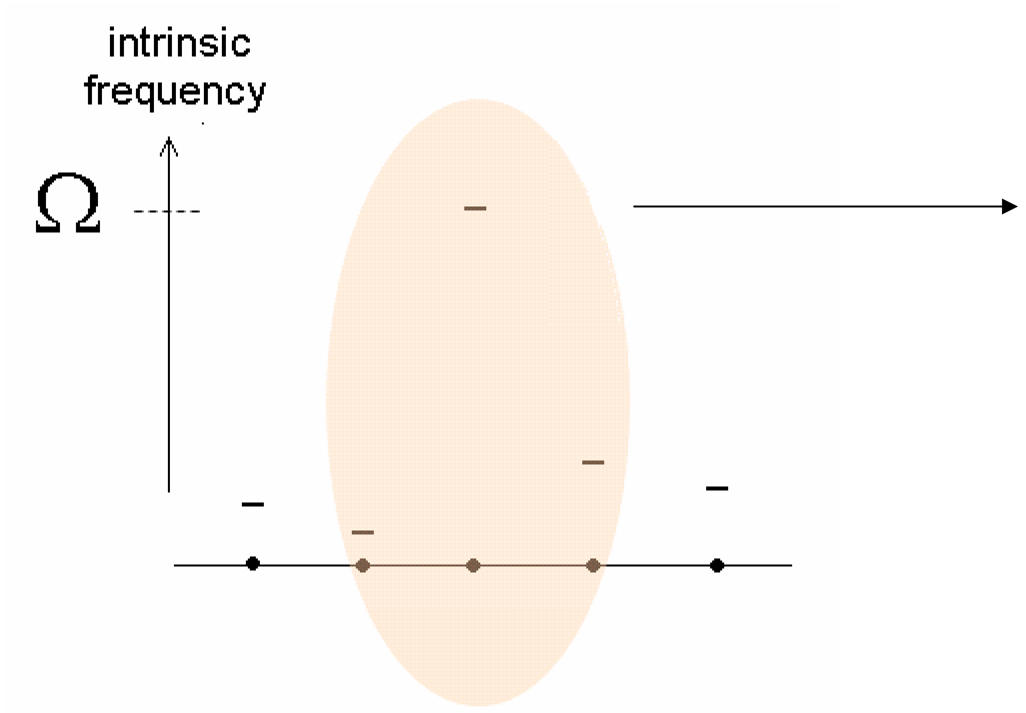
Goal: predict statistical properties of cluster sizes and frequencies.

2 RG steps

(a) “Crazy oscillator” decimation

Crazy oscillator decimation step

$$\dot{\theta}_i = \omega_i + K_{i-1}\sin\left(\theta_{i-1} - \theta_i\right) + K_i \sin\left(\theta_{i+1} - \theta_i\right)$$



$\theta \approx \Omega t$

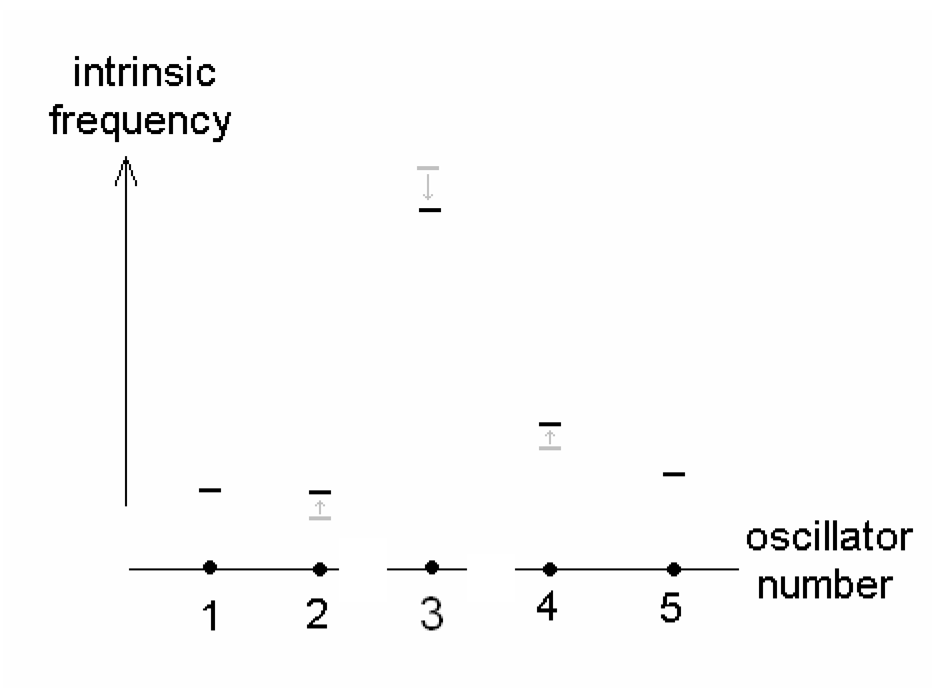
(zeroth-order solution)

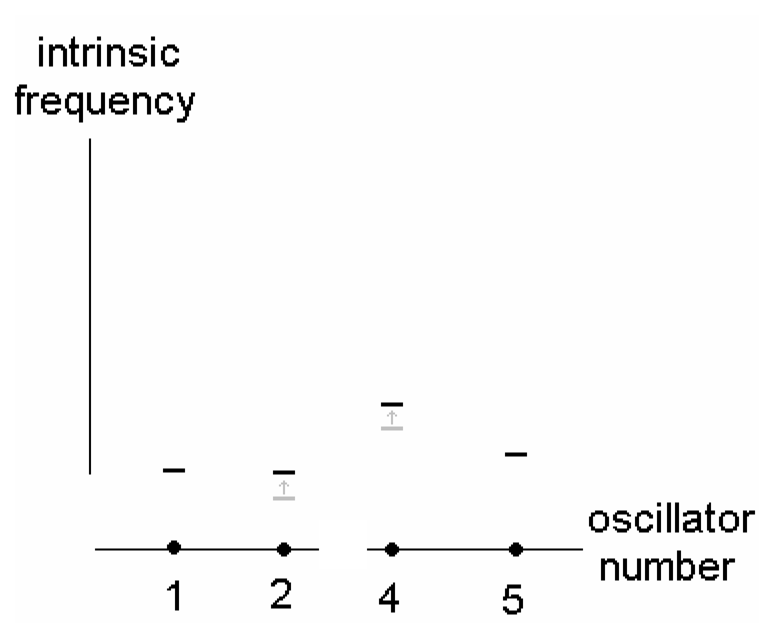
- Strong randomness → $\frac{K}{\Omega}, \frac{\omega}{\Omega} << 1$

- Include influence of the neighbors perturbatively

Results of perturbative calculation:

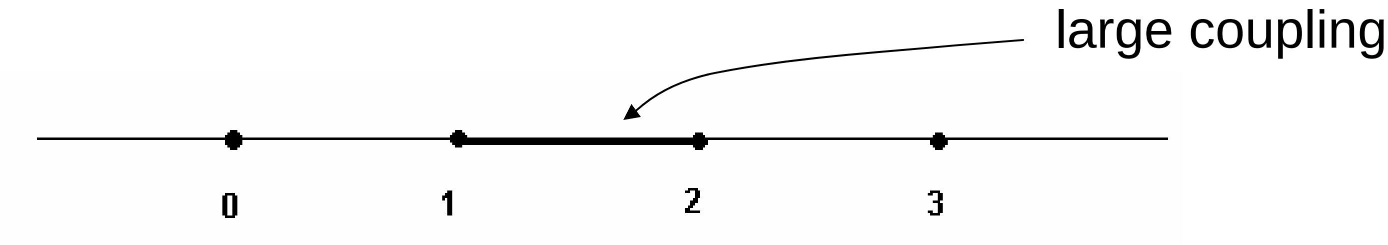
- Frequencies of neighbors get updated: $\omega_2^{'} = \omega_2 + \frac{K_2^2}{\Omega}$ (similar for CO)
- Crazy oscillator dynamics → solved
- Interaction between oscillators 2 and 4 is introduced: $\frac{K_2 K_3}{\Omega}\cos(\theta_2 - \theta_4)$
- Non- synchronizing → no bearing on cluster formation → ignore





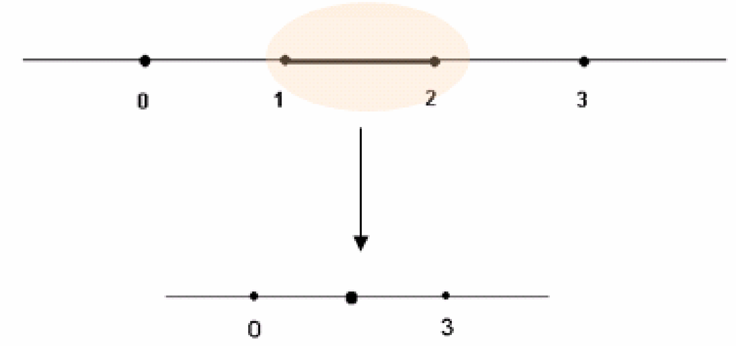
(b) Strong coupling decimation

Strong coupling decimation step:



- If no perturbation → oscillators 1 and 2 lock in phase when $2K > \Delta\omega$
- Strong randomness → $\frac{K_{12}}{\omega_i}, \frac{K_{12}}{K_i} >> 1$ → weak perturbation
- Weak perturbation → phase difference wobbles, but does not grow, i.e. frequencies are still locked
- Perturbative calculation can justify when this is so
- Approximate phase difference by a constant δ

Strong coupling decimation step:

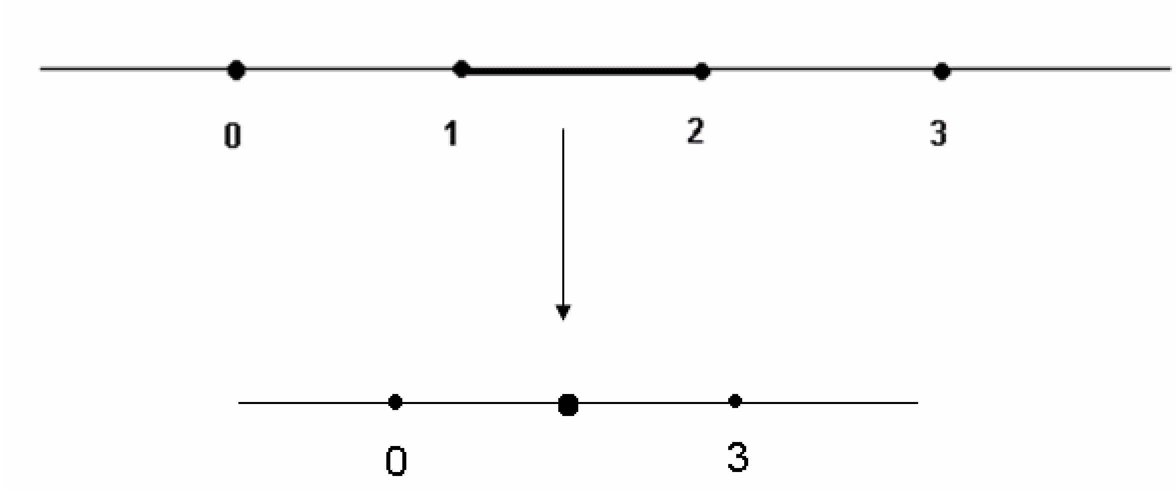


$$m_1\dot{\theta}_1 = m_1\omega_1 + K_1\sin(\theta_0-\theta_1) + K_2\sin(\theta_2-\theta_1)$$
$$m_2\dot{\theta}_2 = m_2\omega_2 + K_2\sin(\theta_1-\theta_2) + K_3\sin(\theta_3-\theta_2)$$

$$M\dot{\Theta} = M\overline{\omega} + K_1\sin\left(\theta_0 - \Theta + \frac{m_2\delta}{M}\right) + K_3\sin\left(\theta_3 - \Theta - \frac{m_1\delta}{M}\right)$$

In 1 dimension, can get rid of δ by re-defining θ s!

Strong coupling decimation step:



$$\left\{ \begin{array}{c} m \\ \theta_1, \theta_2 \\ \omega \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} M = (m_1 + m_2) \\ \Theta = (m_1\theta_1 + m_2\theta_2)/M \\ \omega' = (m_1\omega_1 + m_2\omega_2)/M \end{array} \right\}$$

Strong coupling: two oscillators → one oscillator with different parameters

3 Numerical RG scheme

- “crazy” elements → strong randomness RG: distributions with wide tails.

- Currently working with Lorentzians for both *K*s and ωs .

$$\rho(x) = \frac{\lambda / \pi}{\lambda^2 + x^2}$$

- Choose $\lambda_\omega = 1$ and vary λ_K

Proceed from largest parameters to smallest:

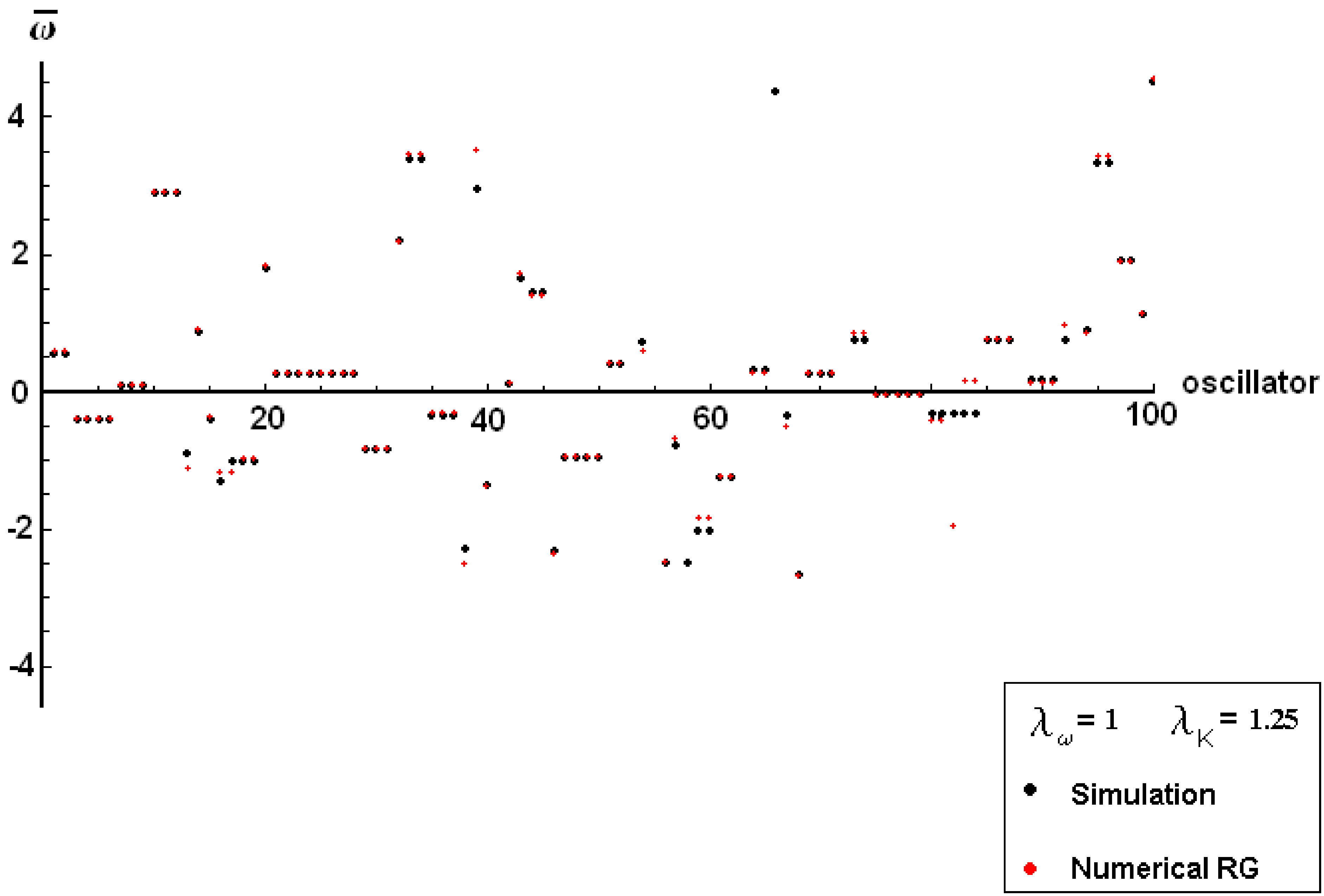
- Largest K ⟶ strong coupling
  - Oscillator pair combined.
  - Mass renormalized: $M = (m_1 + m_2)$
  - Frequency renormalized: $\omega' = (m_1\omega_1 + m_2\omega_2)/M$
- Largest ω ⟶ crazy oscillator (if $K/(\mu\Omega) < 1$)
  - Chain breaks
  - Crazy oscillator frequency renormalized: $\Omega' = \Omega - \frac{K_{1,CO}^2}{2m_1\mu_{1,CO}\Omega} - \frac{K_{3,CO}^2}{2m_3\mu_{3,CO}\Omega}$
  - Neighboring frequency renormalized: $\omega_1' = \omega_1 + \frac{K_{1,CO}^2}{2m_1\mu_{1,CO}\Omega}$

$\mu_{i,j} = \frac{m_i m_i}{m_i + m_j}$

- Decimated-out crazy oscillators model frequency clusters: $\overline{\omega}$ and M
- Repeat until all renormalized oscillators become crazy.

4 Results

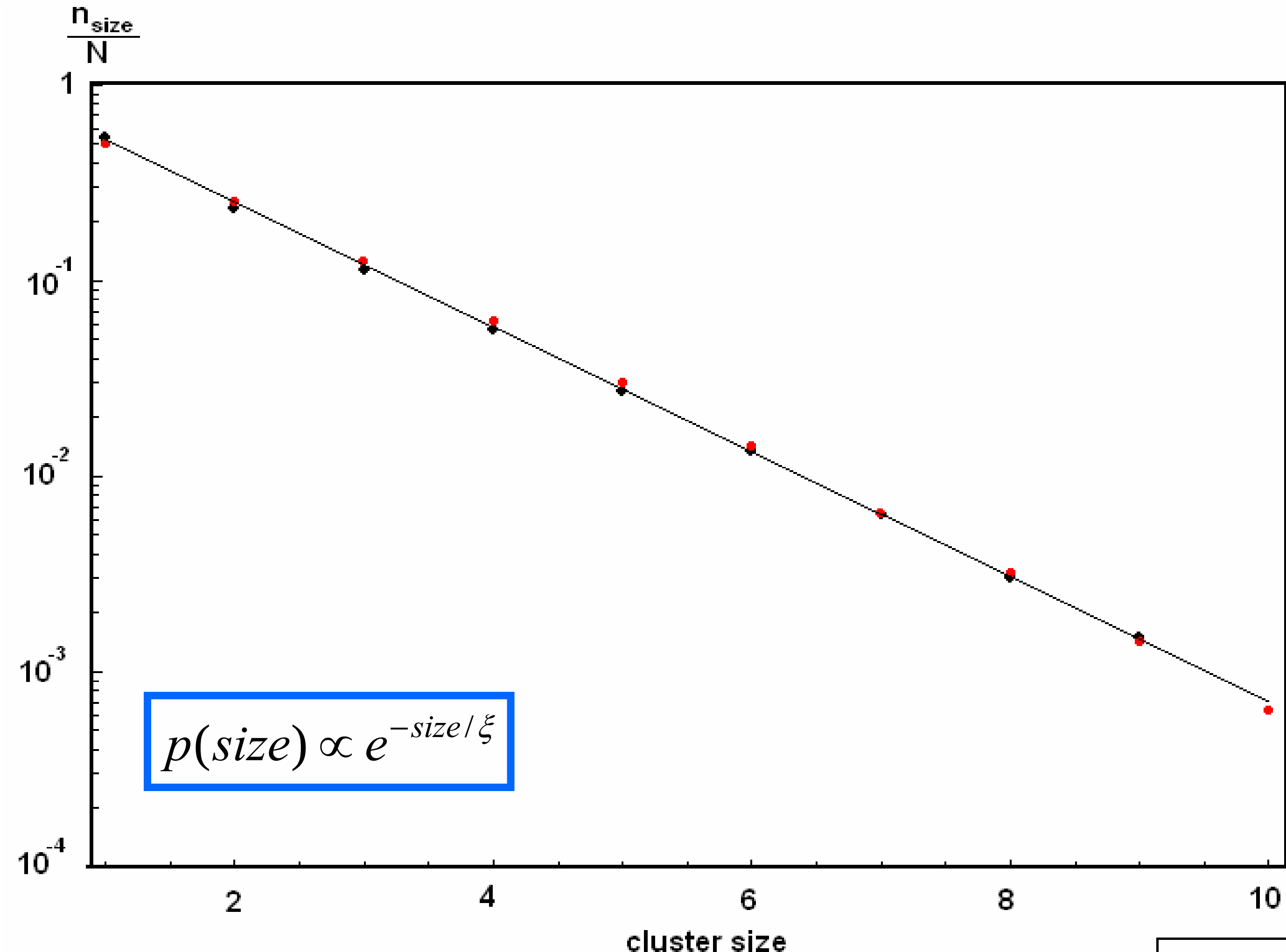
Comparison of cluster frequencies of *simulation* vs. *numerical RG*



RG predicts the clustering effect!

What do we expect for statistics of cluster sizes?

Statistics of cluster sizes: *simulation* vs. *numerical RG*



$\xi \approx 1.4$

- n_{size} = number of **clusters** of a given size
- N = total number of **clusters**
- $\lambda_\omega = 1, \quad \lambda_K = 1.25$

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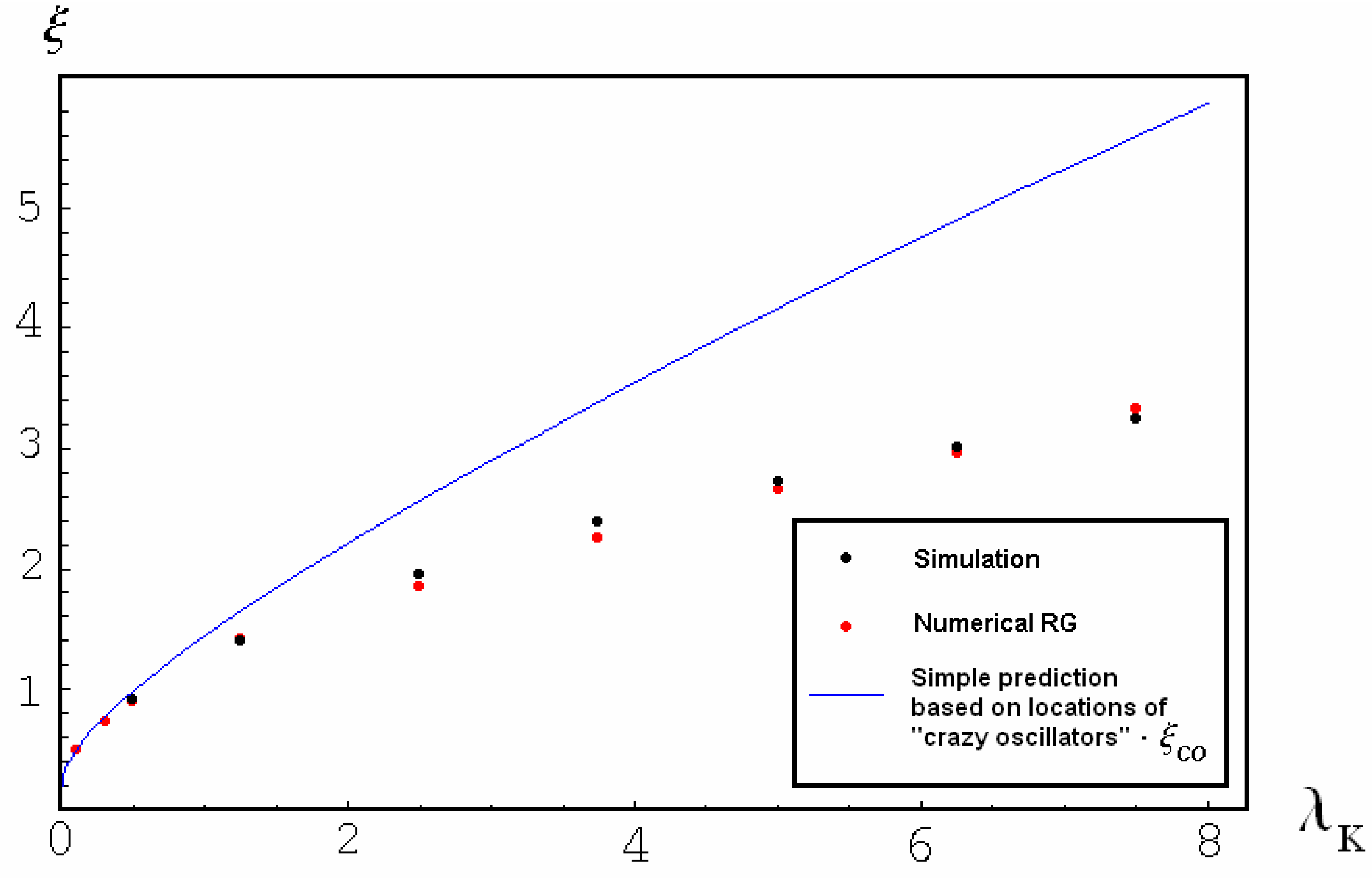
- Simulation
- Numerical RG

Statistics of cluster sizes: *simulation* vs. *numerical RG*

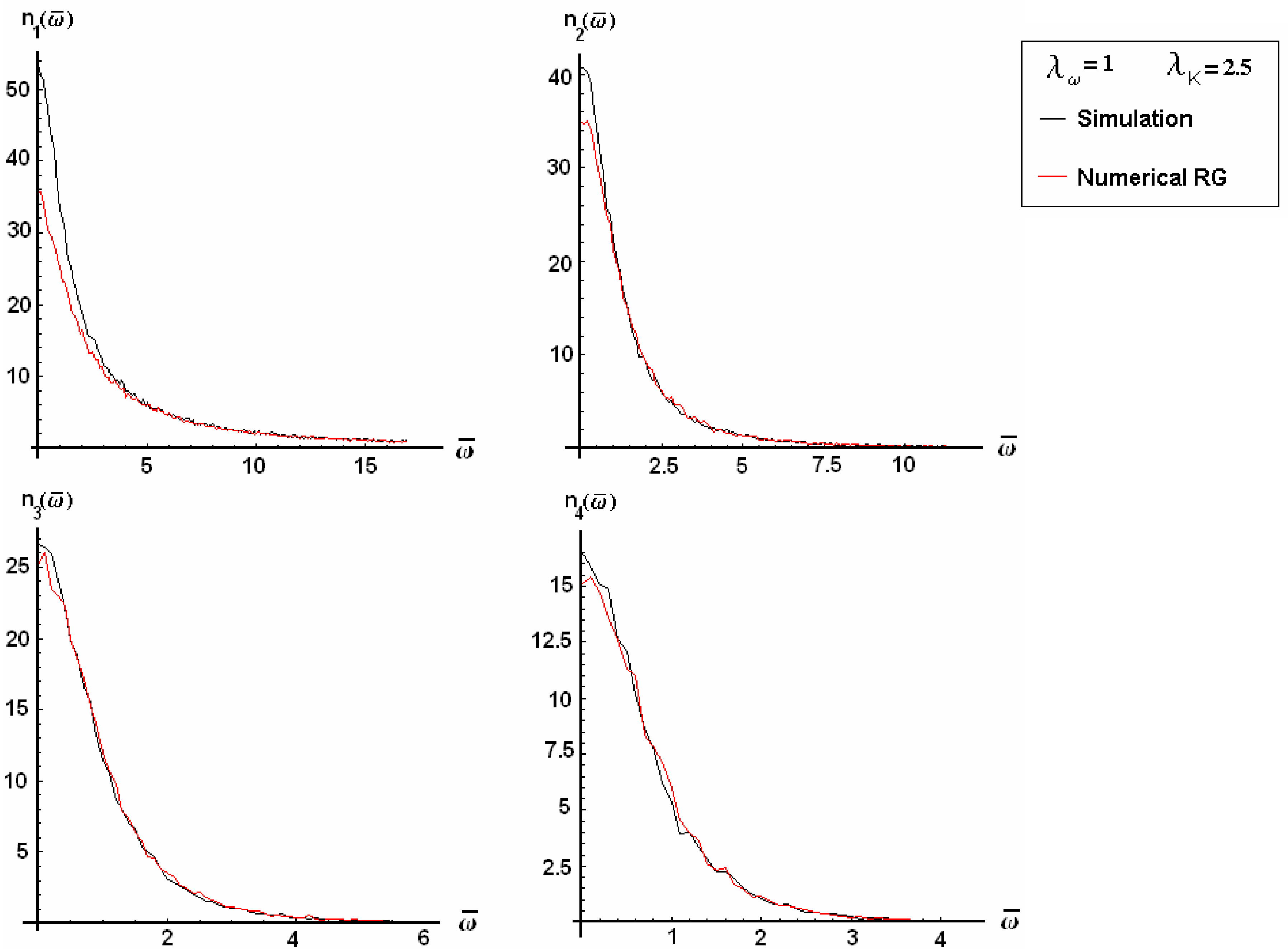


$$\xi_{CO} = -\frac{1}{\ln(1-p)}$$

where *p* is the probability of finding a crazy oscillator based on the distributions of the actual (not renormalized) chain.

Statistics of cluster frequencies

Distribution of cluster frequencies: *simulation* vs. *numerical RG*



Strogatz and Mirollo, 1987:

"The dynamical behavior of

$$\dot{\theta} = \omega_i + K \sum_{j \in \{n.n.\}} \sin\left(\theta_j - \theta_i\right), \qquad i \in \{1,\ldots,l\}^d$$

is not well understood in the regime before phase-locking occurs. In particular, it is not known if or how the distribution of number and size of synchronized clusters scale with K, N, and d".

Method works. More to do …

- Extending this technique to regimes of weaker randomness.
- Analytical RG flow.
- Comparison with probabilitstic points of view:
  - Eample: what is the probability of forming a cluster between two crazy oscillators?
- General dimension.