Resonant forcing of chaotic dynamics

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Overview

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- Current work

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  Szalai, R., Stepan, G., Hogan, S.J. Chaos 14(4), 1069 (2004)
- Host-parasitoid population models

  e.g. Murdoch, W.W., Reeve, J.D. Oikos 50(1), 137 (1987)

Forcing and control

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Example: using parasitoids to control pest insect population

Image courtesy Galveston County Master Gardener Association, Inc

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- We define the response R^2 as the deviation from the unperturbed dynamics:

$$R^2 \equiv \left(\mathbf{x}^{(N)} - \mathbf{y}^{(N)}\right)^2$$

where $\mathbf{y}^{(n+1)} = \mathbf{f}\big(\mathbf{y}^{(n)}\big)$ with $\mathbf{y}^{(0)} = \mathbf{x}^{(0)}$.

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  where $\mathbf{y}^{(n+1)} = \mathbf{f}\big(\mathbf{y}^{(n)}\big)$ with $\mathbf{y}^{(0)} = \mathbf{x}^{(0)}$.
- Constraint: fixed forcing magnitude

$$F^2 = \sum_{n=0}^{N-1} \left(\mathbf{F}^{(n)}\right)^2$$

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- $x_{d_u+1}, \ldots, x_d$ are forced
- $F_i^{(n)} = 0$ for $i = 1, \ldots, d_u$ and $n = 0, 1, \ldots, N-1$.
- If $d_u = 0$ then the problem reduces to the simpler case where all degrees of freedom are forced. [Foster, G., Hübler, A.W., Dahmen, K. Phys. Rev. E 75, 036212 (2007)]

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We seek stationary points of L corresponding to $\partial L / \partial x_i^{(n)} = 0$ and $\partial L / \partial F_i^{(n)} = 0$ for all n and $i = 1, \ldots, d$.

We define

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Then we eliminate $\mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_d^{(n)}$ to obtain equations of motion:

$$\begin{aligned} \left(\mathbf{J}^{(n+1)}\right)^T \mathbf{G}^{(n+1)} &= \mathbf{G}^{(n)} \\ \mathbf{x}^{(N)} - \mathbf{y}^{(N)} &= \lambda \mathbf{G}^{(N-1)}. \end{aligned}$$

Weak forcing

In the case of weak forcing, we Taylor expand the equation of state for small F and obtain this relation:

$$\left(\mathbf{J}^{(n+1)}\right)^T \mathbf{G}^{(n+1)} = \mathbf{G}^{(n)},$$

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where I is the identity matrix,

$$M \equiv I + \sum_{n=1}^{N-1} J^{(N-1)} \cdots J^{(N-n)} \left(J^{(N-n)}\right)^T \cdots \left(J^{(N-1)}\right)^T,$$
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We can solve this system to determine the optimal forcing for any time!

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- $\{\gamma_1^{(n)}, \dots \gamma_{d_u}^{(n)}\}$ are the effective forcing experienced by the degrees of freedom j for which $F_j^{(n)} = 0$
- $\mu^{(n)} = -\frac{R^2}{F^2}\mathbf{G}^{(n)}$ is a product of other Lagrange multipliers and can be eliminated.

Example: coupled shift maps

We consider the mapping function for coupled shift maps:

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} \text{mod}(ax_1^{(n)} + kx_2^{(n)}) \\ \text{mod}(ax_2^{(n)} + kx_1^{(n)}) \end{pmatrix} + \begin{pmatrix} 0 \\ F_2^{(n)} \end{pmatrix}$$

Only x_2 is forced so $d_u = 1$ and $F_1^{(n)} = 0$ for all n.

For $N = 2$ we can solve for the Lagrange multipliers and the optimal forcing function:

$$F_2^{(0)} = -\left(1 - a^2 - k^2 + \beta\right) F_2^{(1)} / 2a,$$

$$F_2^{(1)} = 2aF / \sqrt{4a^2 + \left(1 - a^2 - k^2 + \beta\right)^2},$$

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$$\lambda = \left(1 + a^2 + k^2 - \beta\right) / 2 = \frac{R^2}{F^2}.$$

where

$$\beta \equiv \sqrt{(1 + a^2)^2 + 2k^2(a^2 - 1) + k^4}.$$

We can compare this to the case where both x_1 and x_2 are forced.

$$\tilde{F}_1^{(0)} = -\left(a+k\right)^2 \tilde{F}_2^{(1)}, \qquad \tilde{F}_2^{(0)} = \left(a+k\right)^2 \tilde{F}_2^{(1)},$$

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$$\tilde{\lambda} = 1 + \left(a+k\right)^2 = \frac{\tilde{R}^2}{F^2}.$$

The ratio of final responses

$$\Xi^2 \equiv \frac{\tilde{R}^2}{R^2} = \frac{\tilde{\lambda}}{\lambda} = \frac{2\left[1+\left(a+k\right)^2\right]}{1+a^2+k^2-\beta}$$

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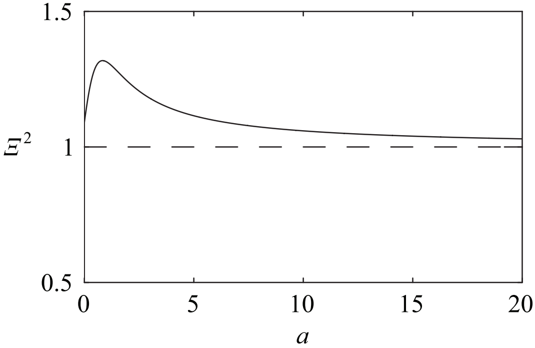
is plotted here for different values of the parameter a.



$F = 0.001$ and $k = 0.3000$.

One test of optimal forcing

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} \mathrm{mod}(a_0 x_1^{(n)} + k x_2^{(n)}) \\ \mathrm{mod}(a_0 x_2^{(n)} + k x_1^{(n)}) \end{pmatrix} + \begin{pmatrix} 0 \\ F_2^{(n)}(a) \end{pmatrix}$$

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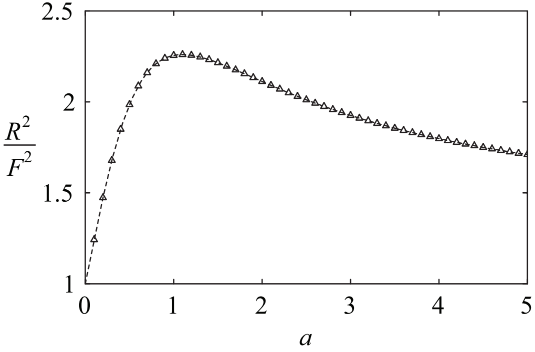
Claim: Unless $a = a_0$, $F_2^{(n)}(a)$ is NOT the optimal forcing function for this system. Thus the response $\frac{R^2}{F^2}$ as a function of a will be maximum at $a = a_0$.



$F = 0.001$, $a_0 = 1.1000$, $k = 0.3000$, $x_1^{(0)} = x_2^{(0)} = 0.1000$. Solid line: analytical result; triangles: numerical calculation. This is not a sufficient condition that we have found the optimal forcing function but it is a necessary one.

Example: one dimensional Hénon map with delay

The forced Hénon map with delay

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In this case we can only force 1 degree of freedom!

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Approximation: To obtain an analytical solution we use

$$x_2^{(1)} \approx y_2^{(1)} = 1 + cx_1^{(0)} - a\left[x_2^{(0)}\right]^2.$$

We are also able to solve the exact system numerically.

Approximate solution is relatively simple:

$$F_2^{(0)} = \left(1 - b^2 - \alpha^2 - \beta\right) F_2^{(1)} / 2\alpha,$$

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where

$$\alpha \equiv 2a\left[1 + c x_1^{(0)} - a\left(x_2^{(0)}\right)^2\right],$$
$$\beta \equiv \sqrt{b^4 + 2b^2\left(\alpha^2 - 1\right) + \left(1 + \alpha^2\right)^2}.$$

Now the same test as for the coupled shift maps:

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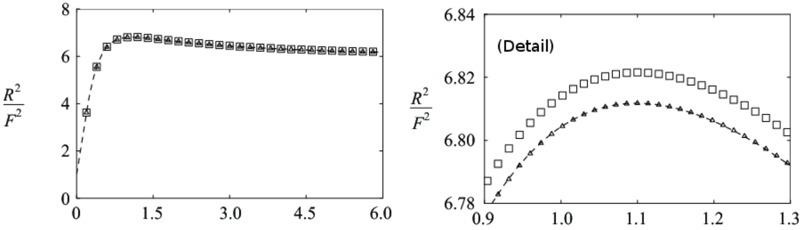
where $F_2^{(n)}(a)$ is the result from the previous page or the result of calculating the exact solution numerically.



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- We demonstrated the method with two examples.
- Forcing only one degree of freedom in a coupled shift map system gives nearly as large a response as forcing both degrees of freedom.
- This method has applications any time a system accurately described by a time-discrete map is to be forced efficiently.

Current work

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- Experimental applications beyond numerical simulations.