From Trajectories to the Ergodic Partition
An Algorithm

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Motivation and purpose

Trajectory plot

Approx. of ergodic partition

Target
Measure-preserving map on a finite/periodic domain.

Goal
Quick, coarse partition of phase space.
Core idea
Ergodic subsets – dynamical \textit{atoms} in phase space.

Why do we care?
\textbf{Analysis} – mapping out phase space
\textbf{Design} – easier to exploit natural dynamics of system

How can we do it?
"Concatenate" trajectories using data clustering methods.

Does our solution measure up?
Yes.
Fast (∼ minutes) two-step algorithm.
Partition corresponds to dynamics in known problems.
Birkhoff’s ergodic theorem

For system $T : \mathcal{M} \to \mathcal{M}$, if $\mathcal{X} \subset \mathcal{M}$ is an ergodic subset, then for $\forall x \in \mathcal{X}, \forall f \in L^1_{\mu}(\mathcal{M})$

- Spatial average
  \[ \bar{f} = \int_{\mathcal{X}} f(x) d\mu(x) \]
- Temporal average
  \[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} f\left(T^k(x)\right) = f^*(x), \]

Level sets of $f^* \to$ invariant partition $\mathcal{P}_f$

Ergodic partition

$\mathcal{P}_E := \bigvee_{f \in L^1_{\mu}} \mathcal{P}_f$

Grouping criterion

$\forall f \in L^1_{\mu}$

$\Rightarrow$ for $x, y \in \mathcal{X}$
On a conceptual level

Algorithm:

1. Choose a *good basis* for observables,
2. Pick a *large* number of ICs in phase space,
3. Simulate system from each IC for an *infinite* time,
4. Compute time averages of observables along trajectories,
5. *Group* trajectories with same time averages into sets.

Limitations:

- No basis in general for $L^1_\mu(\mathcal{M})$, countably infinite for $L^2_\mu(\mathcal{M})$,
- Only finite density of ICs can be chosen,
- Only finite time evolution is computable.

Result: Implementation of grouping criterion unclear.
Implementation
Step 1/2: Simulation

Purpose
Associate time average vector $v_i$ with every trajectory.

Periodic Haar basis

```
\begin{array}{cccc}
0 & 0.25 & 0.5 & 0.75 & 1 \\
1 & 0.75 & 0.5 & 0.25 & 0 \\
\end{array}
```

```
\begin{array}{cccc}
0 & 0.25 & 0.5 & 0.75 & 1 \\
1 & 0.75 & 0.5 & 0.25 & 0 \\
\end{array}
```

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1 & 0.75 & 0.5 & 0.25 & 0 \\
\end{array}
```
Purpose

Implement grouping criterion – assign the same label to similar trajectories.

1. Euclidean graph:
   nodes $\rightarrow$ trajectories
   edges $\rightarrow$ $g_{ij} = ||v_i - v_j||_2$

2. Diffusion distance graph: adds robustness to data distribution

3. Dominant eigenspace of random walk: natural coordinate system

4. CC Clustering: reveals and labels dominant features
\[ J_{n+1} = J_n + \lambda \sin(2\pi \theta_n) \pmod{1} \]
\[ \theta_{n+1} = J_{n+1} + \theta_n \pmod{1} \]
\[ f : S^2 \to \mathbb{R}^n \quad f \in L^2(S^2) \]

- Poincaré map of periodically forced harmonic oscillator
- Measure-preserving; resonant and chaotic zones
- \( \lambda \in (0, 1) \) tunes amount of chaos
- Observables – Haar basis on \( S^2 \)

Unnecessary detail in chaotic region. No obvious way of color-coding regions.
Partition quality analysis
Averaging horizon length

- $\lambda = 0.3; 100[Box/Dim]$
- Iterations (clockwise): 50, 600, 1700
- More iterations:
  - Longer simulation step
  - High spectral ridge (gap)

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Partition quality analysis
Discretization resolution

- $\lambda = 0.3$; 2000 Iterations
- Resolution $[Box/Dim]$ (clockwise): 50, 106, 300
- Higher resolution:
  - Longer clustering step
  - Low spectral ridge (gap)
Running time analysis
Complexity of dynamics

\[ \lambda = 0.10 \]

\[ \lambda = 0.75 \]

- Blue – simulation step
- Red – total running time

Influence of dynamics (\( \lambda \))

Execution time [s]

Simulation Step

Total

Resolution–Performance dependence

Simulation Step

Total

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Running time analysis

Algorithm parameters

More iterations:
- asymptotical behavior more faithfully approximated,
- increased simulation cost,
- reduced clustering cost.

Higher resolution:
- able to resolve finer features,
- increased simulation cost,
- increased clustering cost.
Final remarks

**Achievements**
- **Goal**: valid partitioning
- **Speed**: order of minutes on a laptop
- **Efficiency**: only interesting segments of phase space can be analyzed

**Challenges**
- Tuning of clustering step
- Validation of result for unknown behavior
- Parallelization – necessity for higher dimensions
- Extension to continuous time systems (ODEs/DAEs)

**Conclusion**
Efficient computational tool improvable with further theoretical development.

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Igor Mezić.
On the geometrical and statistical properties of dynamical systems: theory and applications.

Igor Mezić and Stephen Wiggins.
A method for visualization of invariant sets of dynamical systems based on the ergodic partition.

Mikhail Belkin and Partha Niyogi.
Towards a theoretical foundation for laplacian-based manifold methods.

Geometric diffusions as a tool for harmonic analysis and structure definition of data: Diffusion maps.

Zoran Levnajić.
Ergodic partition theory and visualization of invariant sets and resonances in discrete dynamical systems.

Zoran Levnajić and Igor Mezić.
Ergodic theory and visualization i: Visualization of ergodic partition and invariant sets.

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