A review of some dynamical systems problems in plasma physics

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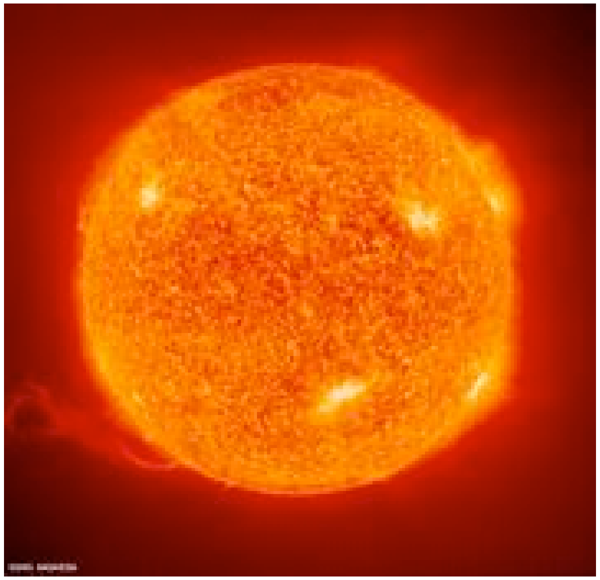
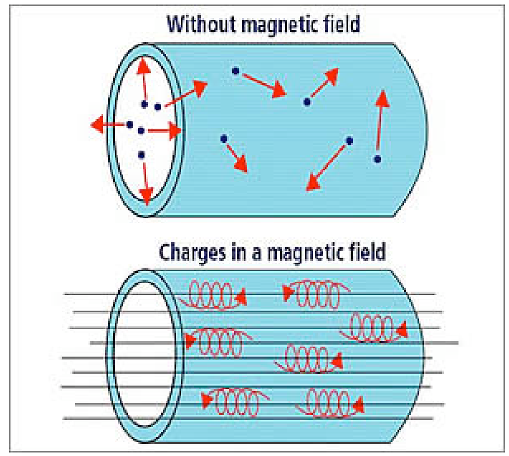
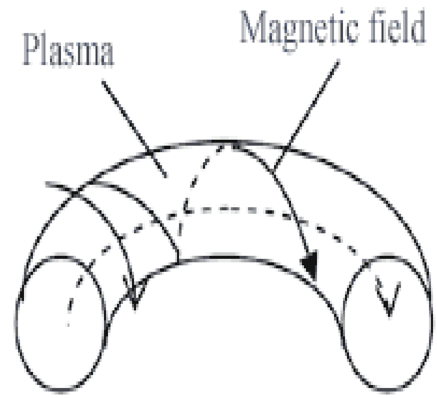
Fusion plasmas

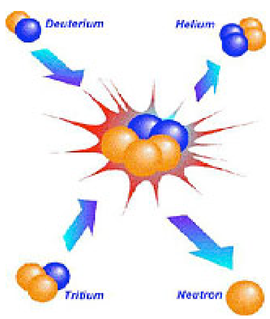
Fusion in the sun

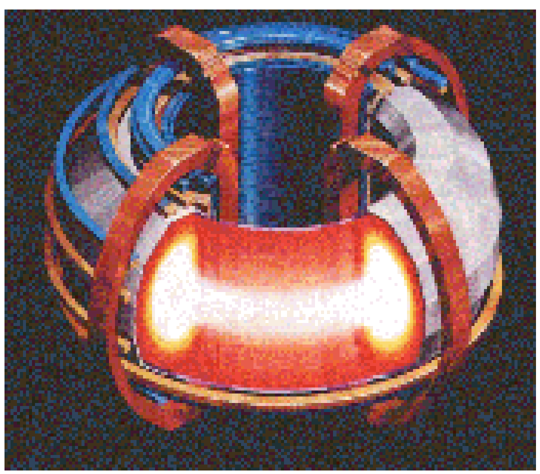


Magnetic confinement









Controlled fusion on earth

- Understanding radial transport is one of the key issues in controlled fusion research
- This is highly non-trivial problem!
- Standard approaches typically underestimate the value of the transport coefficients due to the presence of anomalous diffusion

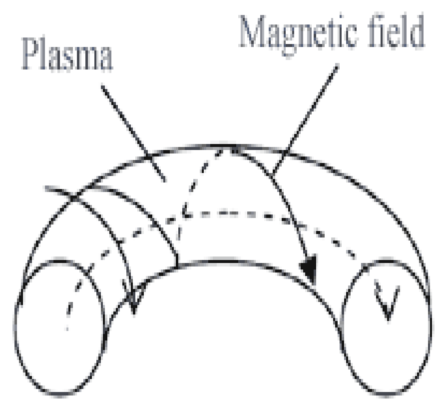
Magnetic field lines
Hamiltonian chaos

Magnetic fields and Hamiltonian systems

Using $\nabla \cdot \vec{B} = 0$ we can write $\vec{B} = \nabla\psi \times \nabla\theta - \nabla\chi \times \nabla\zeta$

and express the field line equations as a Hamiltonian system

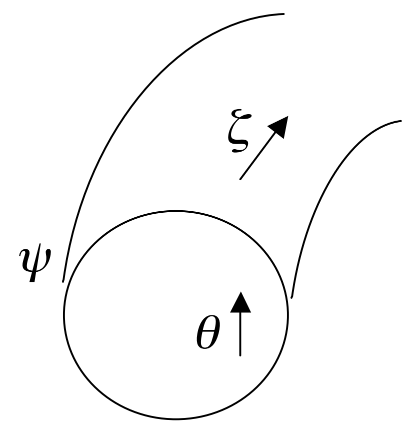
$$\frac{d\psi}{d\zeta} = -\frac{\partial \chi}{\partial \theta} \qquad \frac{d\theta}{d\zeta} = \frac{\partial \chi}{\partial \psi}$$



$\zeta =$ toroidal coordinate $\Rightarrow$ time

$\chi =$ poloidal flux $\Rightarrow$ Hamiltonian

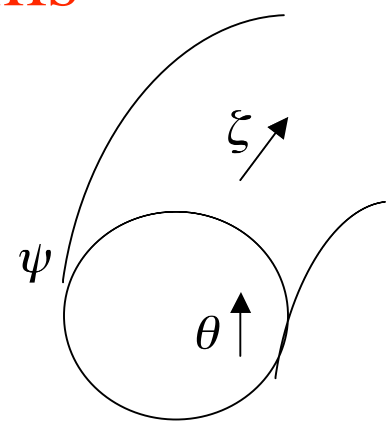
$\left.\begin{array}{l}\psi = \text{flux coordinate} \\ \theta = \text{poloidal coordinate}\end{array}\right\} \Rightarrow$ Canonical conjugate phase space coordinates



Perturbation of integrable systems

$\chi = \chi_0(\psi)$ $\Rightarrow$ Well-defined flux surfaces, integrable system

$$\frac{d\theta}{d\zeta} = \frac{\partial \chi_0}{\partial \psi} = \frac{1}{2\pi}\iota(\psi)$$



$\chi = \chi_0(\psi) + \chi_1(\psi,\theta,\zeta)$ $\Rightarrow$ in general not well-defined flux surfaces, non-integrable system.

The perturbation of integrable systems $H = H_0(J) + H_1(J,\theta,t)$ is *"the fundamental problem of dynamics"* (Poincare)

From a numerical and analytical perspectives, it is convenient to approach this problem in the context of area preserving maps.

$$x_{i+1} = x_i + \Omega(y_{i+1}) + f(x_{i,}, y_{i+1})$$

$$y_{i+1} = y_i + g(x_{i,}, y_{i+1})$$

$x_{i,}$ = angle-like, periodic coordinate

$y_{i,}$ = action-like, radial coordinate

perturbation

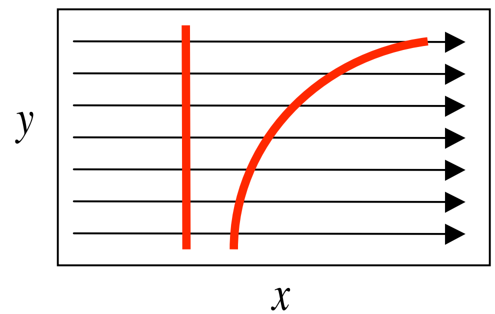
KAM theorem

$$x_{i+1} = x_i + \Omega(y_{i+1}) + f(x_{i,}, y_{i+1})$$

$$y_{i+1} = y_i + g(x_{i,}, y_{i+1})$$

Twist condition

$$\frac{d\Omega}{dy} \geq K > 0$$

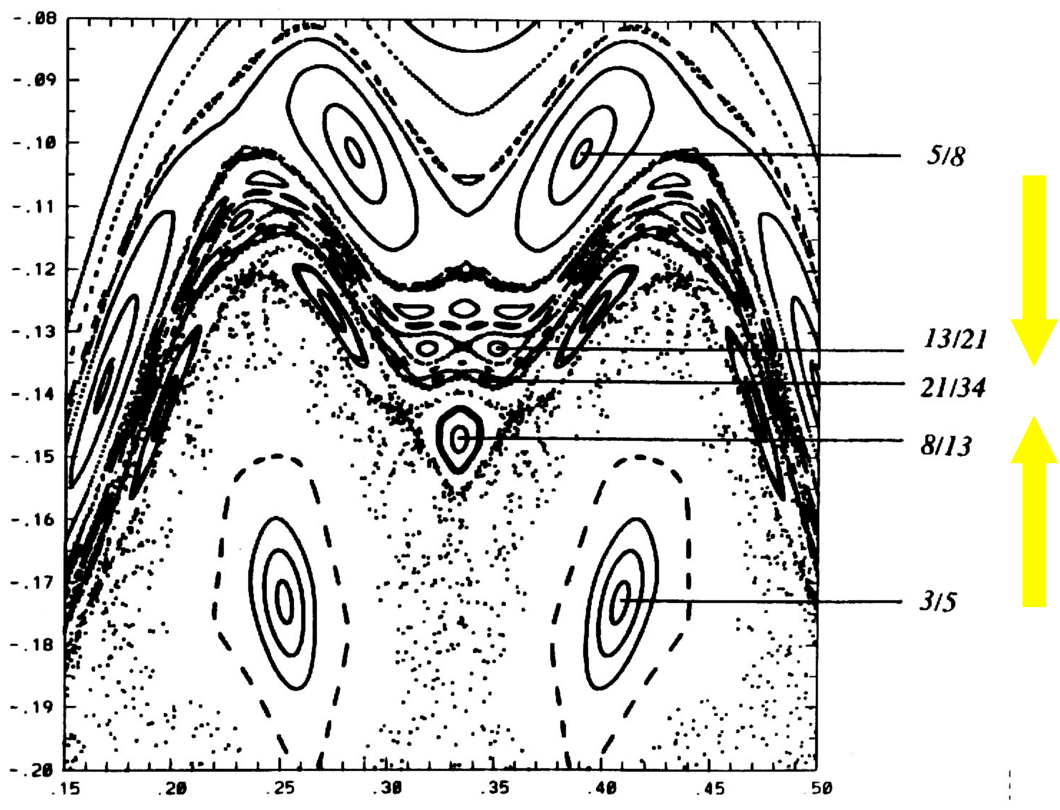


If Ω satisfies the twist condition $\frac{d\Omega}{dy} \geq K > 0$ and it is *j*-times differentiable, then, there is an $\varepsilon > 0$ such that all maps with $|f|_j + |g|_j < \varepsilon K C^2$ have invariant circles for all winding numbers ω satisfying $\left|\omega - \frac{m}{n}\right| > \frac{C}{n^\tau}$ for any any *(m,n)* with $2 < \tau < (j+1)/2$

Periodic orbits approximation

Periodic orbits allow us to locate (target) specific magnetic flux surfaces.

$$\underbrace{\frac{1}{2},\frac{2}{3},\frac{3}{5},\frac{5}{8},\frac{8}{13},\ldots\ldots\frac{F_{n-1}}{F_n}\ldots\ldots}_{\text{convergents of continued fraction expansion}} \rightarrow \underbrace{\iota_{\min}=\frac{\sqrt{5}-1}{2}}_{\text{rotational transform of target torus}}$$



In the same way that the convergents F_{n-1}/F_n approach ι_0 the F_{n-1}/F_n periodic orbits bracket the torus.

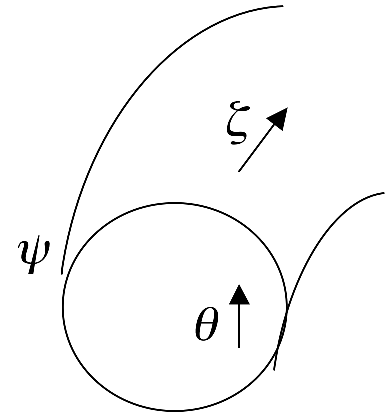
Non-twist systems and reversed shear

Integrable case

$$\chi = \chi_0(\psi)$$

$$\frac{d\psi}{d\zeta} = 0$$

$$\frac{d\theta}{d\zeta} = \frac{\partial \chi_0}{\partial \psi} = \frac{1}{2\pi}\iota(\psi)$$



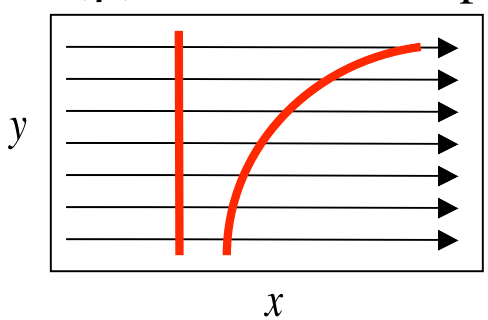
Corresponding map

$$x_{i+1} = x_i + \Omega(y_{i+1})$$

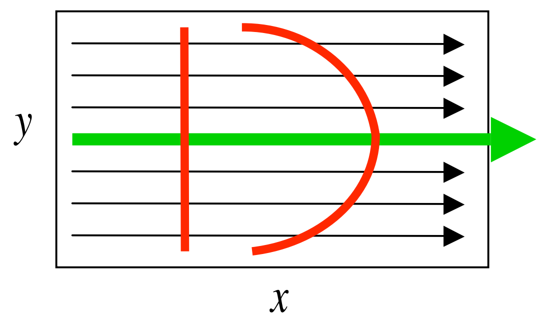
$$y_{i+1} = y_i$$

Monotonic $\iota(\psi)$ → twist map

$$\frac{\partial x_{i+1}}{\partial y_i} \neq 0$$



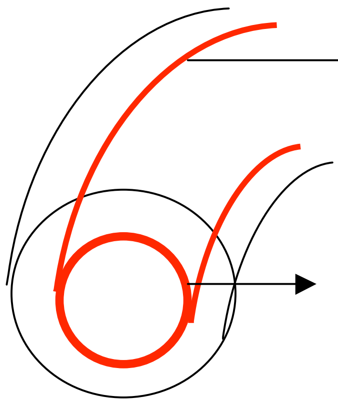
Reversed shear $\iota'(\psi) = 0$ → non-twist map



$$\frac{\partial x_{i+1}}{\partial y_i} = 0$$

Reversed shear configurations violate the twist condition and the KAM theorem can not be applied to them.

Shearless torus problem



$$\frac{d\iota}{d\psi} = 0$$

$$\iota_{\min} = \frac{\sqrt{5}-1}{2}$$

•The goal is to study the stability of the shearless torus with a given fixed value of rotational transform as function of the perturbation amplitude.

•KAM theorem guarantees the robustness of irrational tori with shear. But in the absence of shear the theorem can not be applied.

•This problem of direct practical interest to compact stellarators.

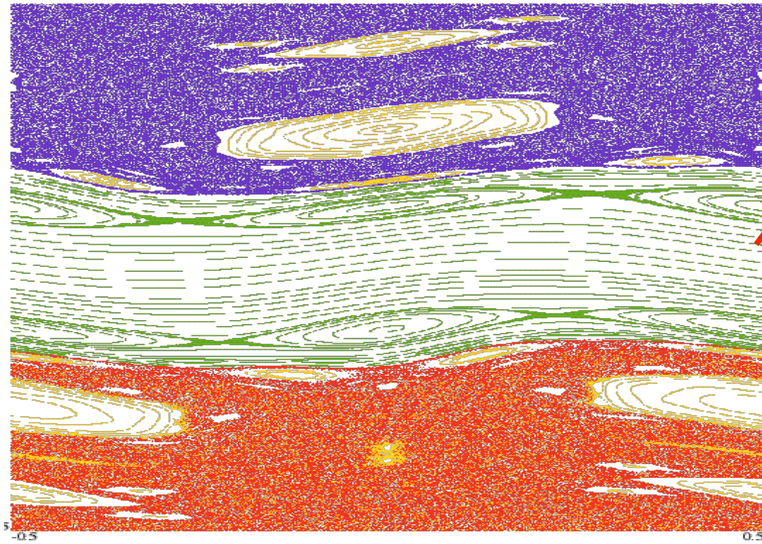
•This is a fusion plasmas motivated problem that has opened a new research direction in dynamical systems.

•This problem is also of interest in the study of transport in shear flows (jets) in the geophysical fluid dynamics.

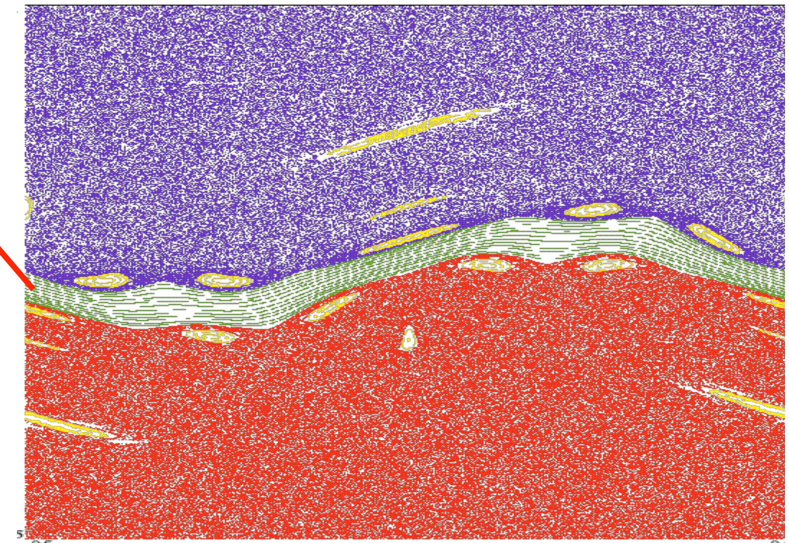
del-Castillo-Negrete, D., J.M. Greene, and P.J. Morrison: Physica D, 91, 1-23, (1996); Physica D, 100, 311-329, (1997)

del-Castillo-Negrete, D., and P.J. Morrison: Phys. Fluids A, **5**, 948-965, (1993).

Non-twist map below threshold of shearless torus destruction

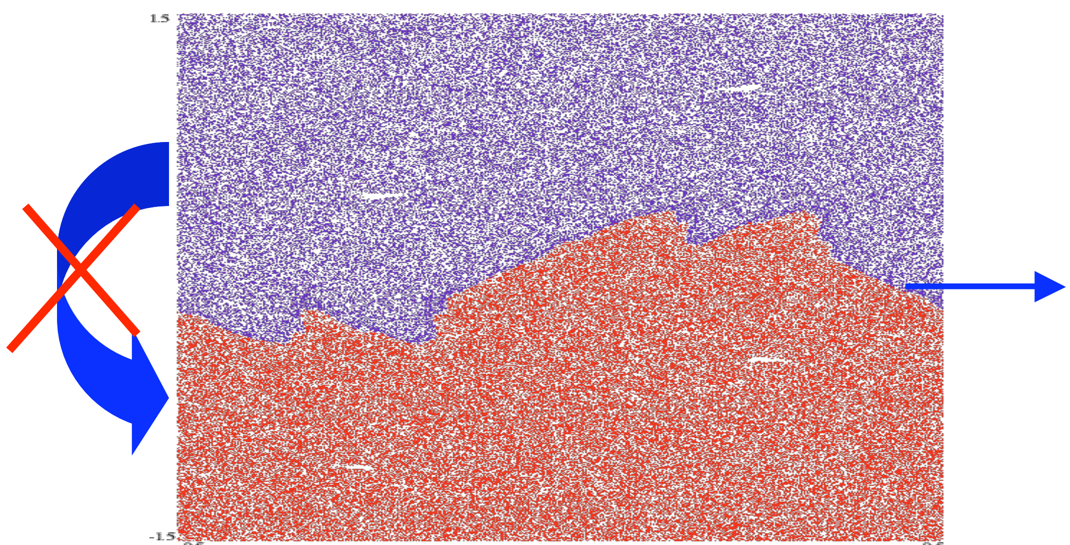


shearles
torus



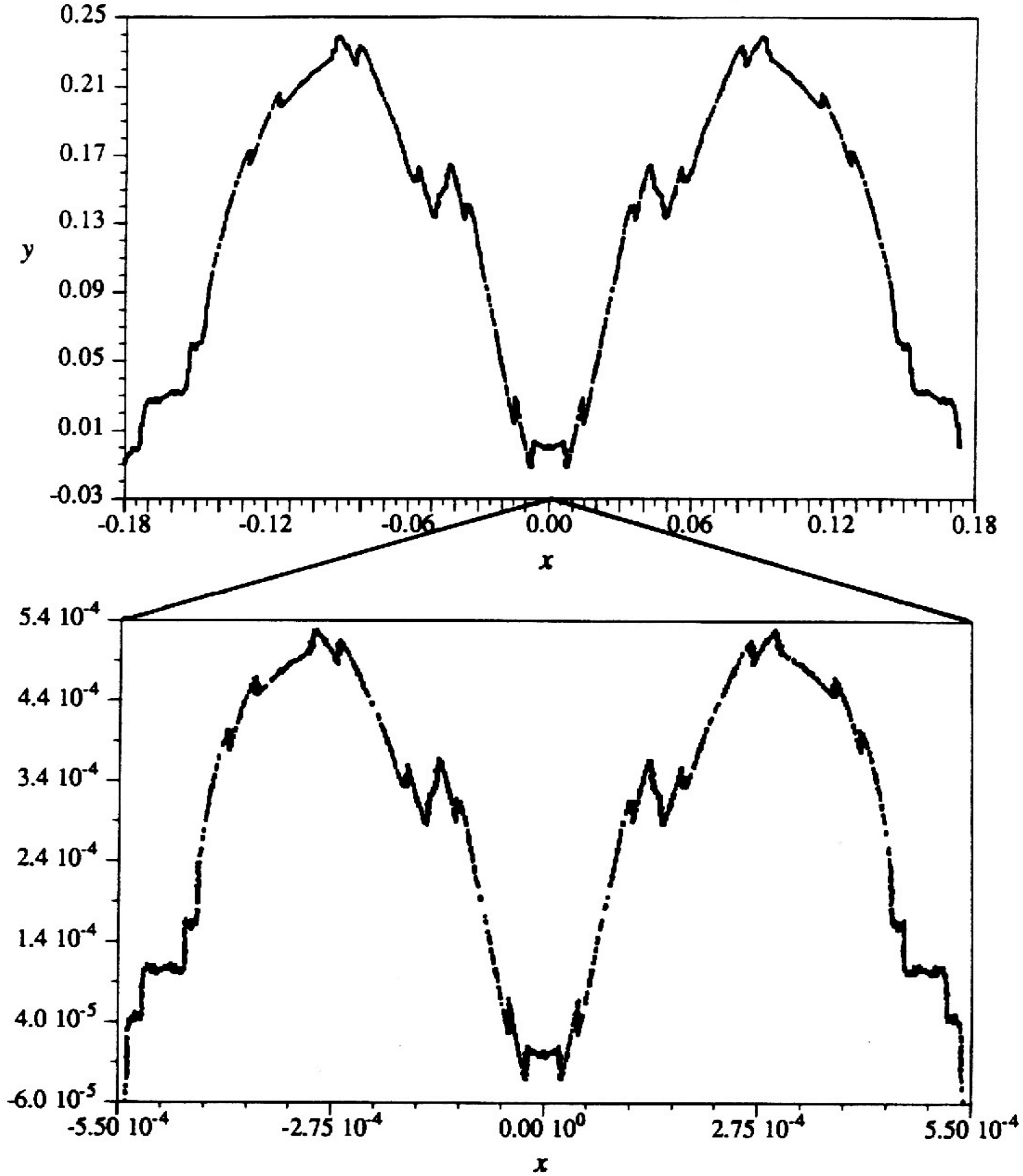
Non-twist map at the onset of shearless torus destruction $\lambda < \lambda_c$

no mixing



Shearless torus
magnetic
confinement
barrier

Fractal structure of shearless torus at criticality



$$(x', y') = (\alpha x, \beta y)$$

Universal scale factors

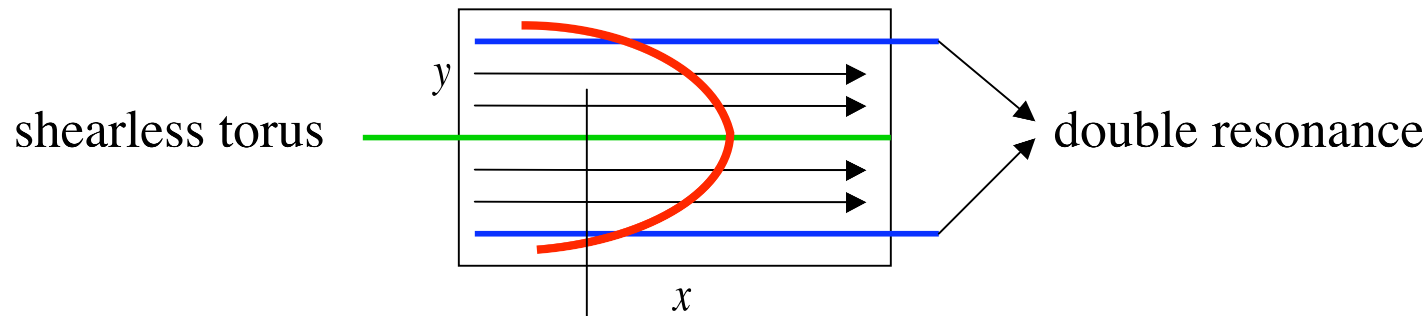
$\alpha = 322$

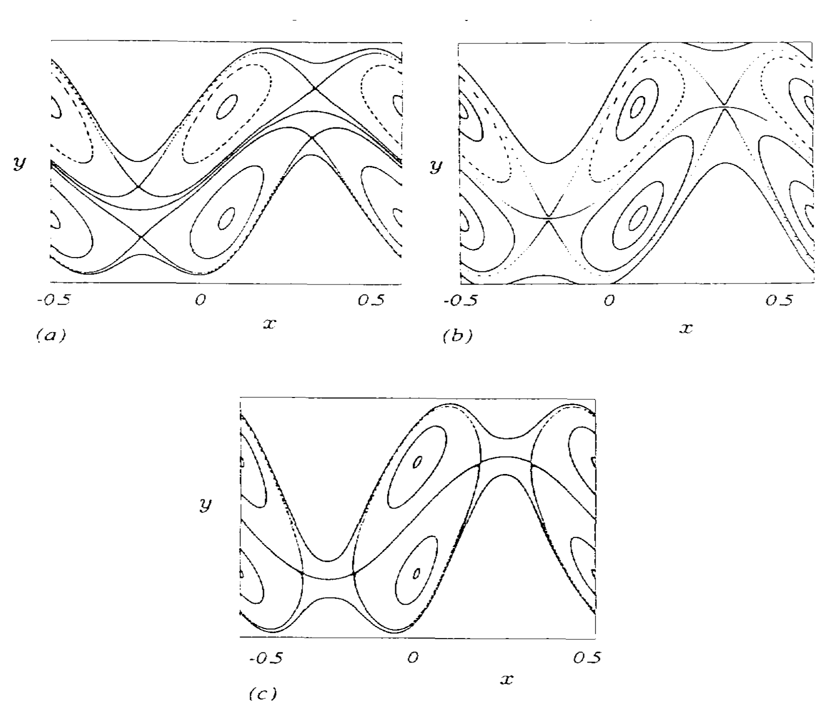
$\beta = 464$

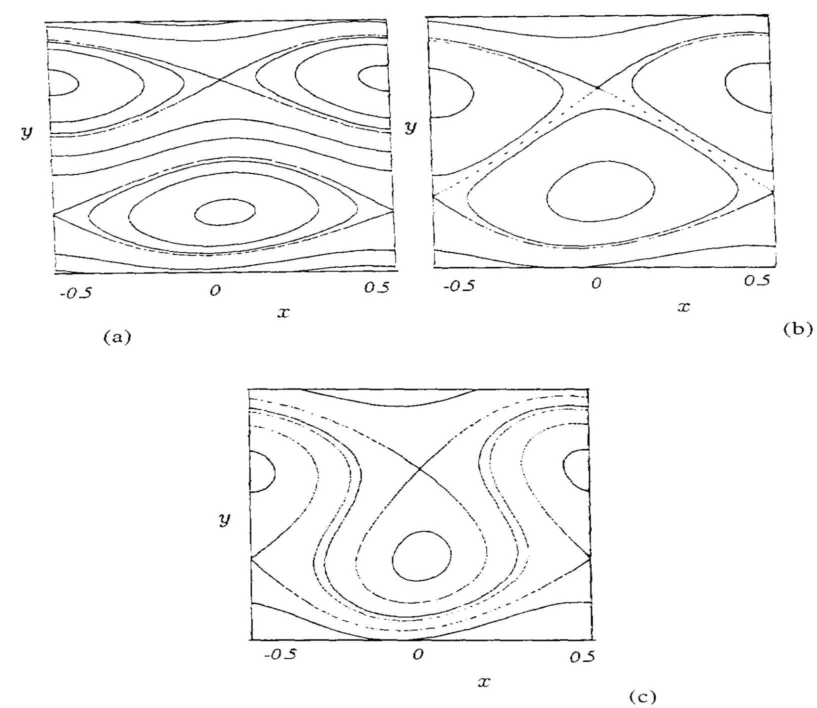
The transition to chaos in non-twist systems defines a new universality class for the transition to chaos in Hamiltonian systems

Separatrix reconnection

A signature of non-twist maps is separatrix reconnection which is a generic change in the phase space topology due to the violation of the twist condition.

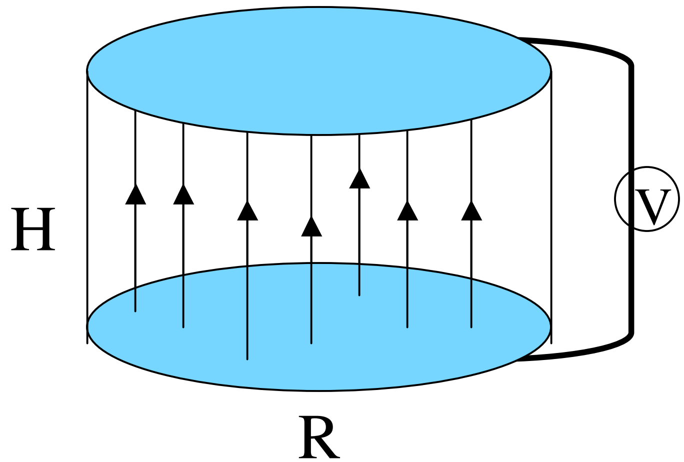






Magnetic field self-organization and Chaotic scattering

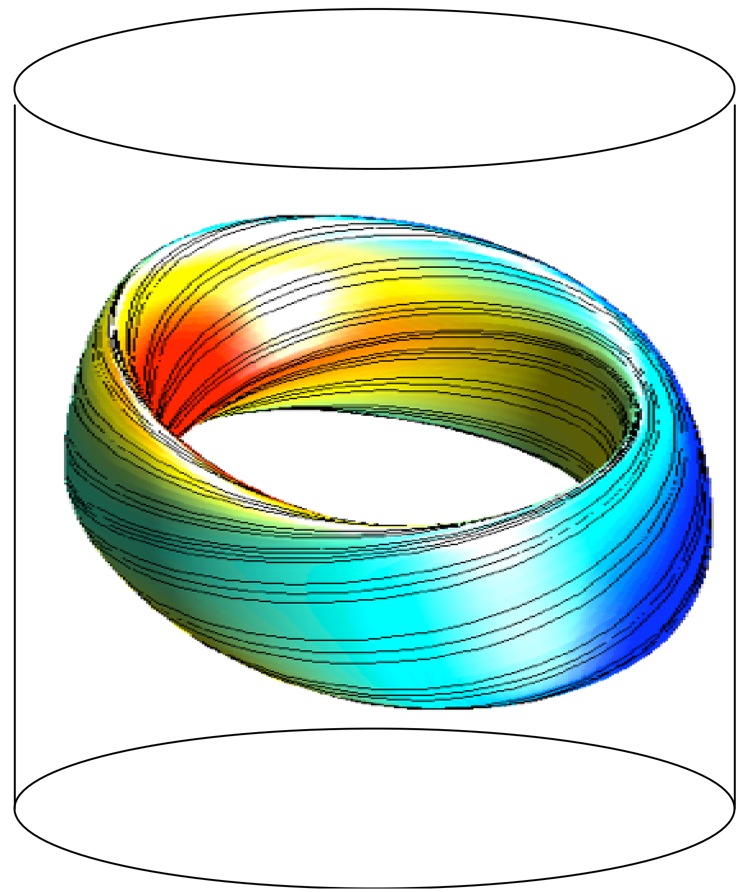
Spheromak formation



•Cylindrical geometry

•Perfectly conducting electrodes

•Fixed magnetic flux at the electrodes



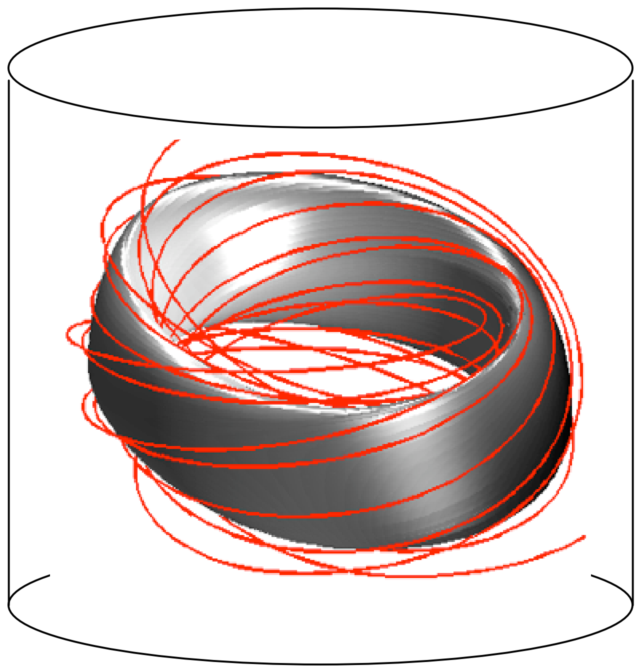
•Resistive, zero beta, finite viscosity,
•constant density, three dimensional MHD

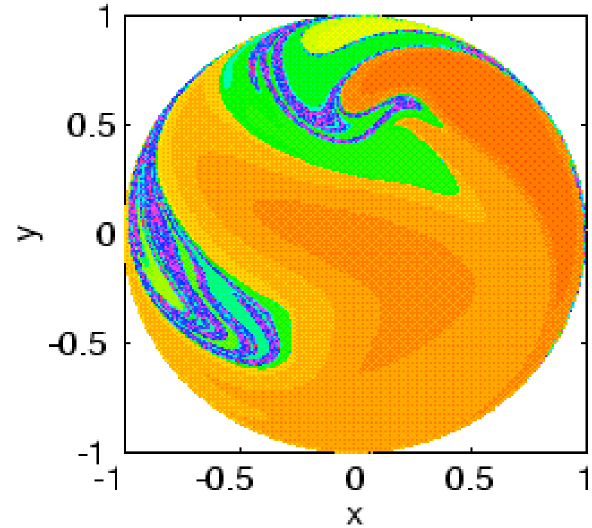
$$\rho\left(\partial_t\,\vec{v}+\vec{v}\cdot\nabla\vec{v}\right)=\vec{j}\times\vec{B}+\mu\nabla^2\vec{v}$$

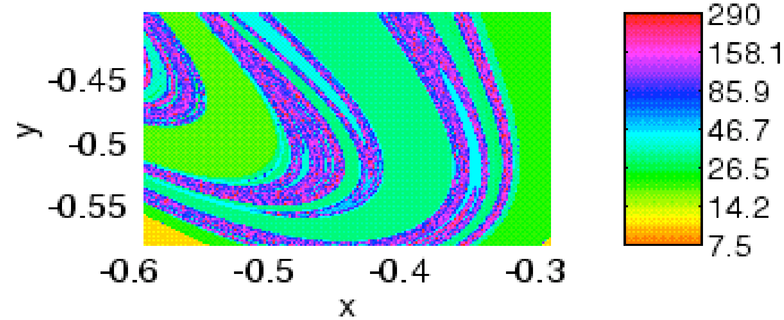
$$\vec{E}+\vec{v}\times\vec{B}=\eta\,\vec{j}$$

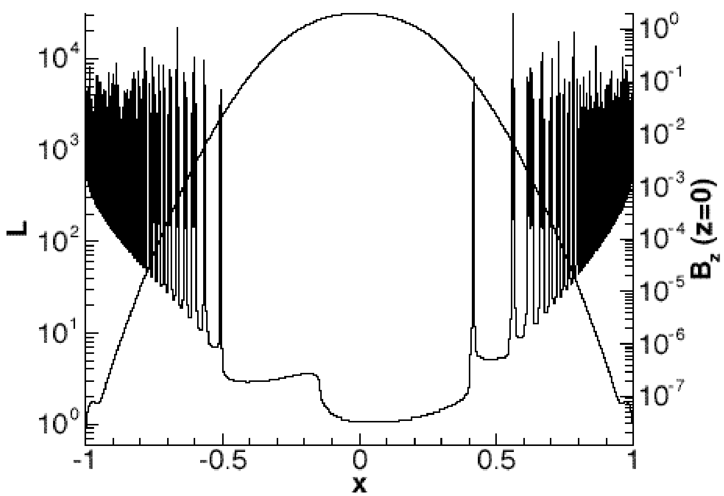
C. Sovinec, Finn, J.~M., and D. del-Castillo-Negrete:
Physics of Plasmas, 8, (2), 475-490, (2001).

Chaotic scattering









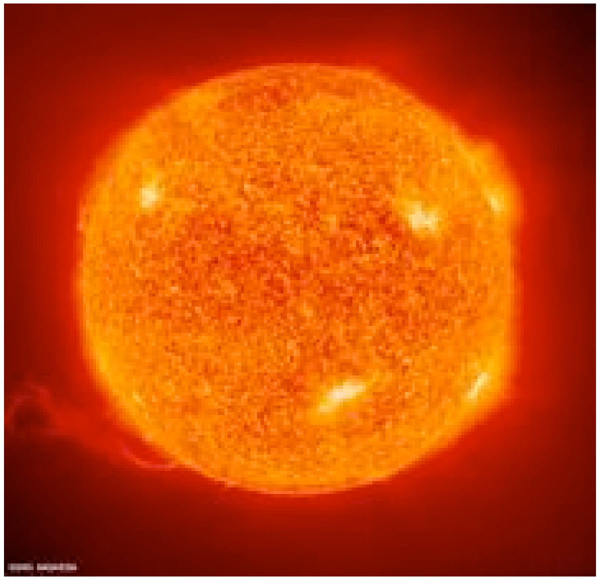
- The field line length L is the analogue of the time delay function in chaotic scattering.
- L exhibits the typical fractal like behavior of chaotic scattering

Finn, J.M., C. Sovinec, and D. del-Castillo-Negrete: Phys. Rev. Lett. 85, (21), 4538-4541, (2000).

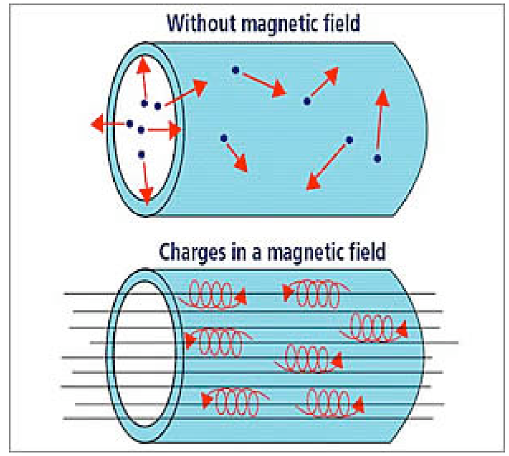
Self-consistent chaos and globally coupled oscillators

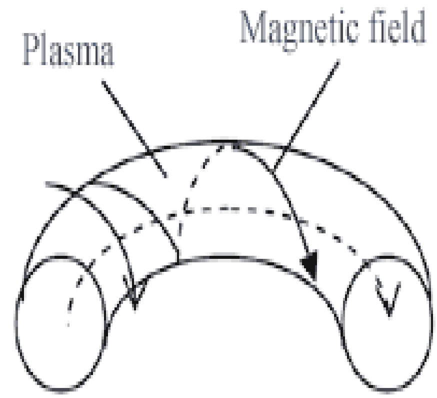
Fusion plasmas

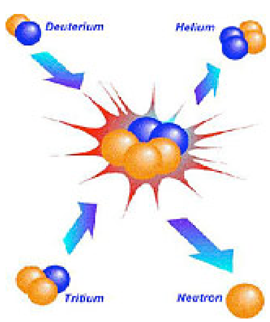
Fusion in the sun

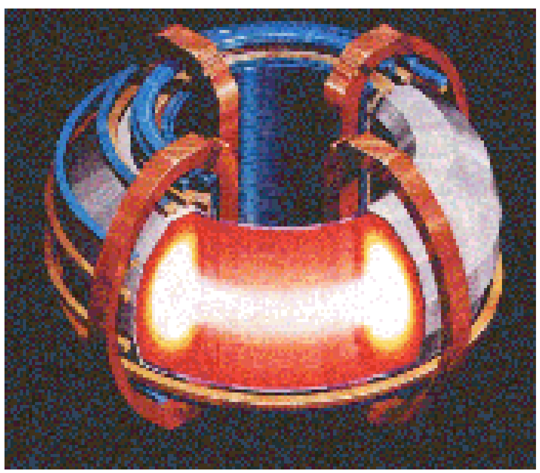


Magnetic confinement





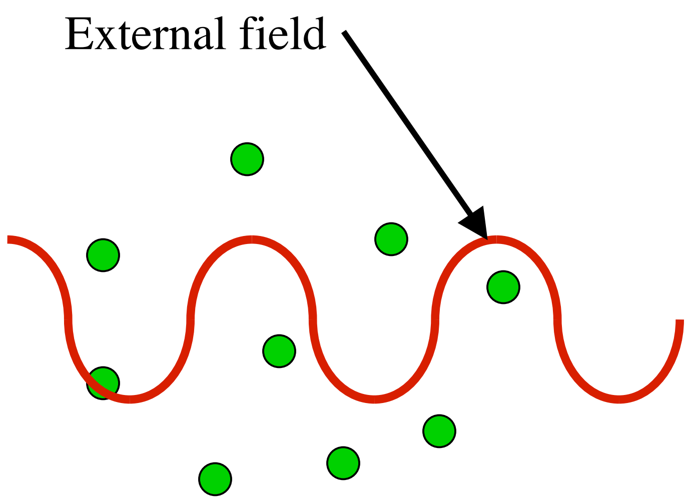




Controlled fusion on earth

- Understanding radial transport is one of the key issues in controlled fusion research
- This is highly non-trivial problem!
- Standard approaches typically underestimate the value of the transport coefficients due to the presence of anomalous diffusion

Simplest Hamiltonian problem



Example: particle in an external electrostatic field

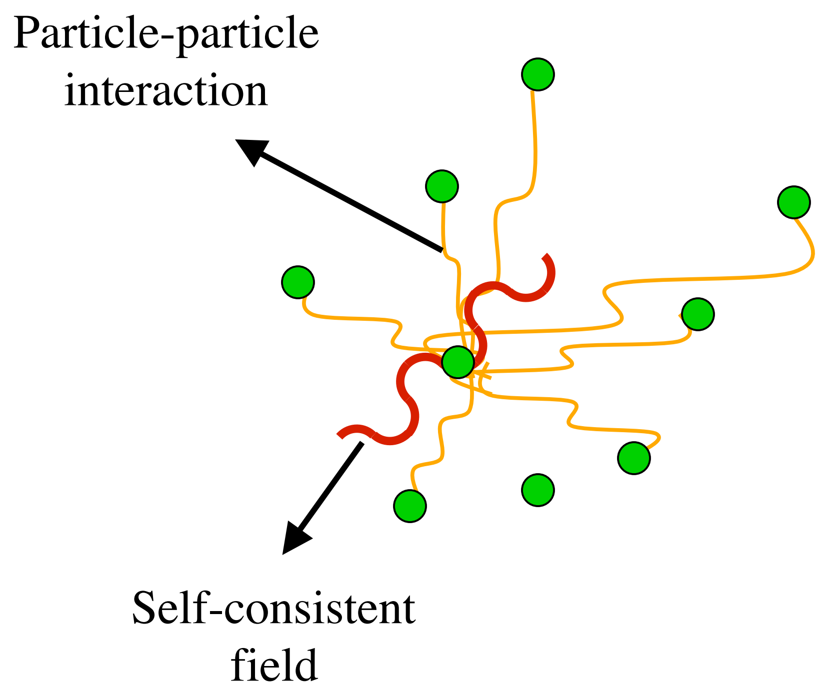
$$\frac{d\,q_j}{dt} = p_j$$

$$\frac{d\,p_j}{dt} = \cos(q_j - \omega t)$$

No coupling between particles, low dimensional phase space

$$H = \sum_{j=1}^{N} \left[\frac{p_j}{2} - \sin(q_j - \omega t) \right]$$

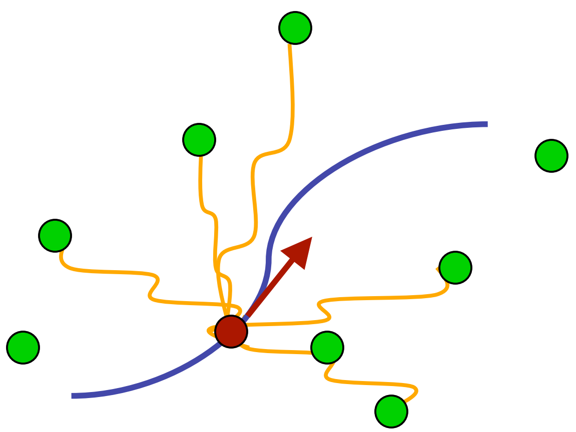
N-body problem



Fully coupled dynamical system, large number of degrees of freedom, high dimensional phase space

$$H = H(q_1, \cdots q_N; p_1, \cdots p_N)$$

Self-consistent one-dimensional electron dynamics



$$\frac{dx}{dt} = \frac{\partial H}{\partial u} \qquad \frac{du}{dt} = -\frac{\partial H}{\partial x}$$

$$H = \frac{u^2}{2} - \phi(x,t)$$

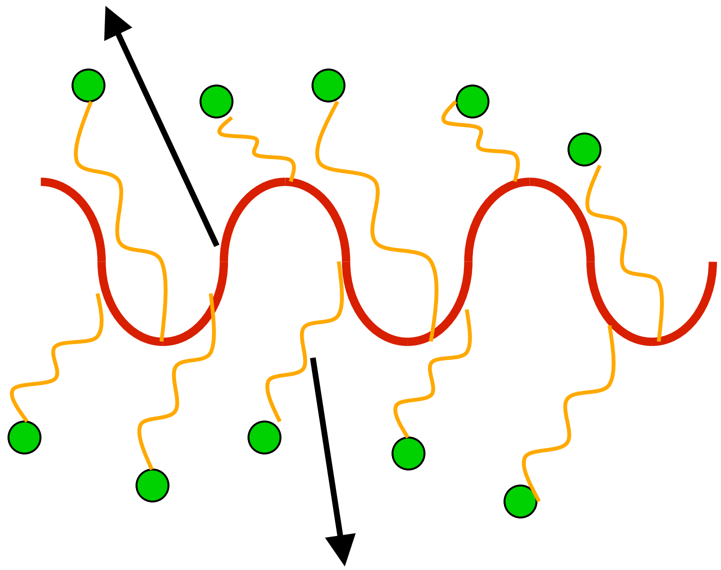
$$\nabla^2 \phi = \int f\, du - \rho_i$$

Poisson equation

$$\phi(x,t) = \int dx'\, G(x;x') \int du'\, f(x',u',t)$$

Mean-field problem

Mean-field



Mean-field--particle interaction

Particle dynamics:

$$\frac{d\,z_j}{dt} = F(z_j, \phi)$$

Mean-field dynamics:

$$\frac{d\,\varphi}{dt} = G\left(\varphi\,; z_1, z_1, \ldots z_N\right)$$

Problem of intermediate complexity. Like in the test particle problem, all particles are described by the same dynamical system. This system depends on a mean field whose evolution is determined by the N-body coupling.

Mean field model

$$\frac{d\,x_j}{dt} = y_j \qquad j = 1,2,\ldots\ldots N$$

$$\frac{dy_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

particles

$$\underbrace{\frac{d}{dt}\left(\rho\,\mathrm{e}^{-i\theta}\right) + i\,U\rho\,\mathrm{e}^{-i\theta} = i\sum_k \Gamma_k\, e^{-ix_k}}_{\text{Mean field}}$$

Hamiltonian formulation

$$H = \sum_{j=1}^{N}\left[\frac{1}{2\Gamma_j}p_j^2 - 2\Gamma_j\sqrt{J}\cos(x_j - \theta)\right]$$

$$a = \sqrt{J}\,e^{-i\theta} \qquad p_k = \Gamma_k\,y_k$$

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial p_k} \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial x_k}$$

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial J} \qquad \frac{dJ}{dt} = -\frac{\partial H}{\partial \theta}$$

$\Gamma_k > 0$ Clump

$\Gamma_k < 0$ Hole

O'Neil, Winfrey &Malmberg, (1971)
del-Castillo-Negrete, D., Phys. of Plasmas, **5**, (11), 3886-3900, (1998); CHAOS, **10**, (1), 75-88, (2000); Plasma Physics and Controlled Fusion, **47**, 1-11 (2005).

Analogies with coupled oscillator models

Phase coupled oscillators models

$$\dot{\phi}_k = \omega_k + \frac{K}{N}\sum_j \sin\left[\phi_j - \phi_k\right]$$

$$\dot{\phi}_j = \omega_j - \rho(t)\sin\left[\phi_j - \theta(t)\right]$$

$$\rho\, \mathrm{e}^{\mathrm{i}\theta} = \frac{K}{N}\sum_k e^{-i\phi_k}$$

Single wave model

$$\ddot{x}_j = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

$$-i\frac{d}{dt}\left(\rho\, \mathrm{e}^{\text{-i}\theta}\right) + U\rho\, \mathrm{e}^{\text{-i}\theta} = \sum_k \Gamma_k\, e^{-ix_k}$$

Mean field Hamiltonian X-Y model

$$\ddot{\phi}_k = \frac{K}{N}\sum_j \sin\left[\phi_j - \phi_k\right]$$

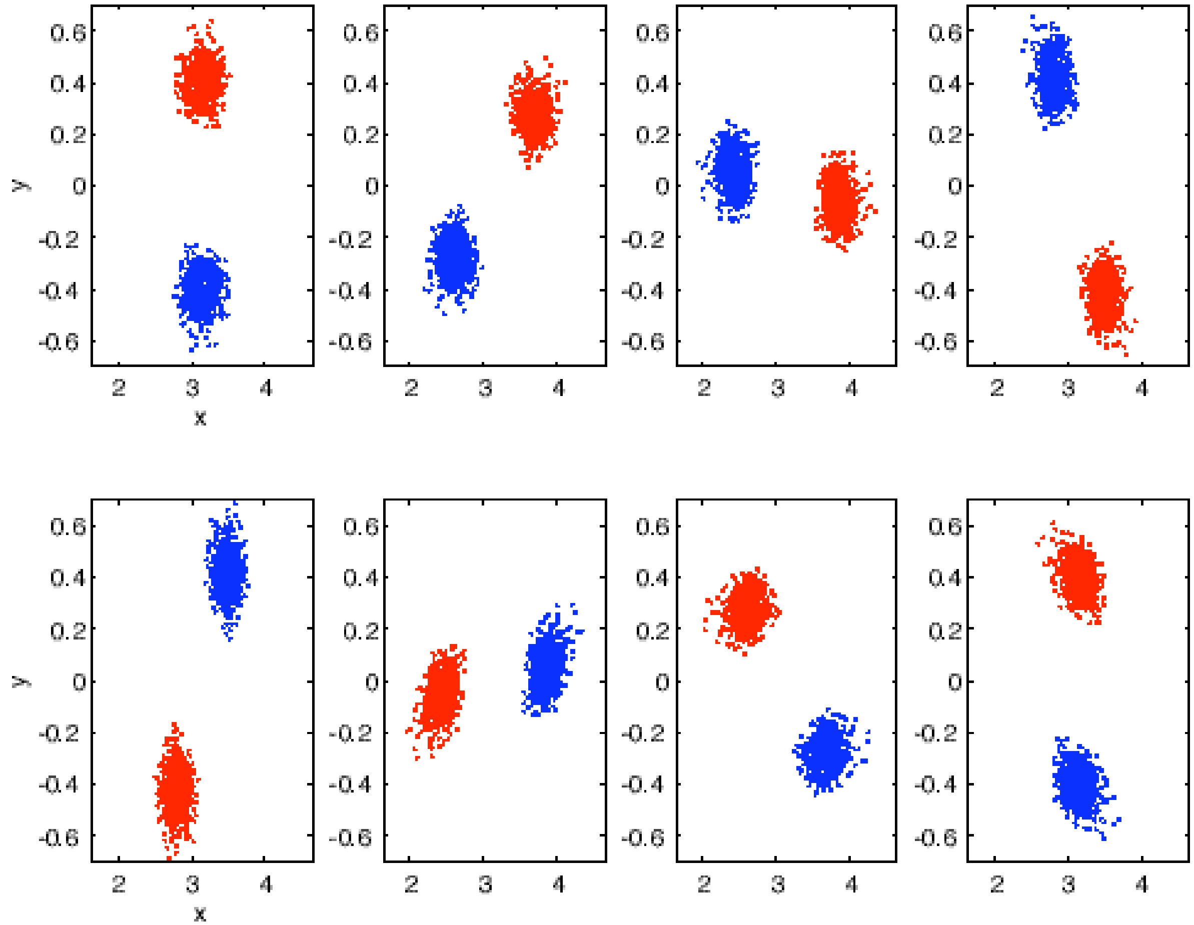
$$\ddot{\phi}_k = -\rho(t)\sin\left[\phi_k - \theta(t)\right]$$

$$\rho\, \mathrm{e}^{\mathrm{i}\theta} = \frac{K}{N}\sum_j e^{i\phi_j}$$

D. del-Castillo-Negrete,
"Dynamics and self-consistent chaos in a mean field HamiltonianModel". Chapter in Dynamics and Thermodynamics of Systems with Long Range Interactions, T. Dauxois, et al. Eds.
Lecture Notes in Physics Vol. 602, Springer (2002).

Coherent dipole rotation
Finite--N particle simulation

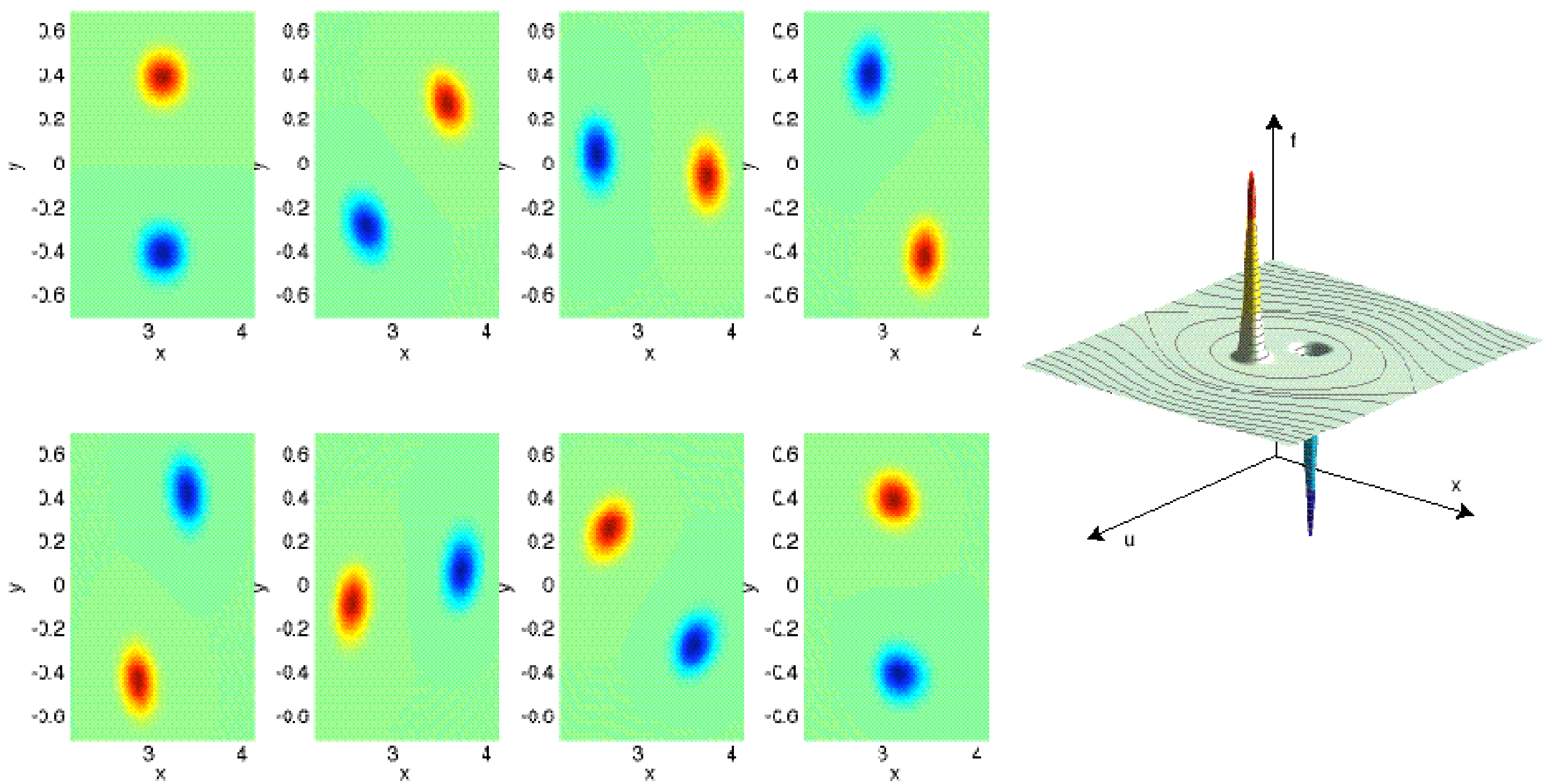
N=1000



Red: clump
Blue: hole

del-Castillo-Negrete, M.C. Firpo, CHAOS, **12**, 496 (2002).

Coherent dipole rotation
Infinite--N kinetic simulation



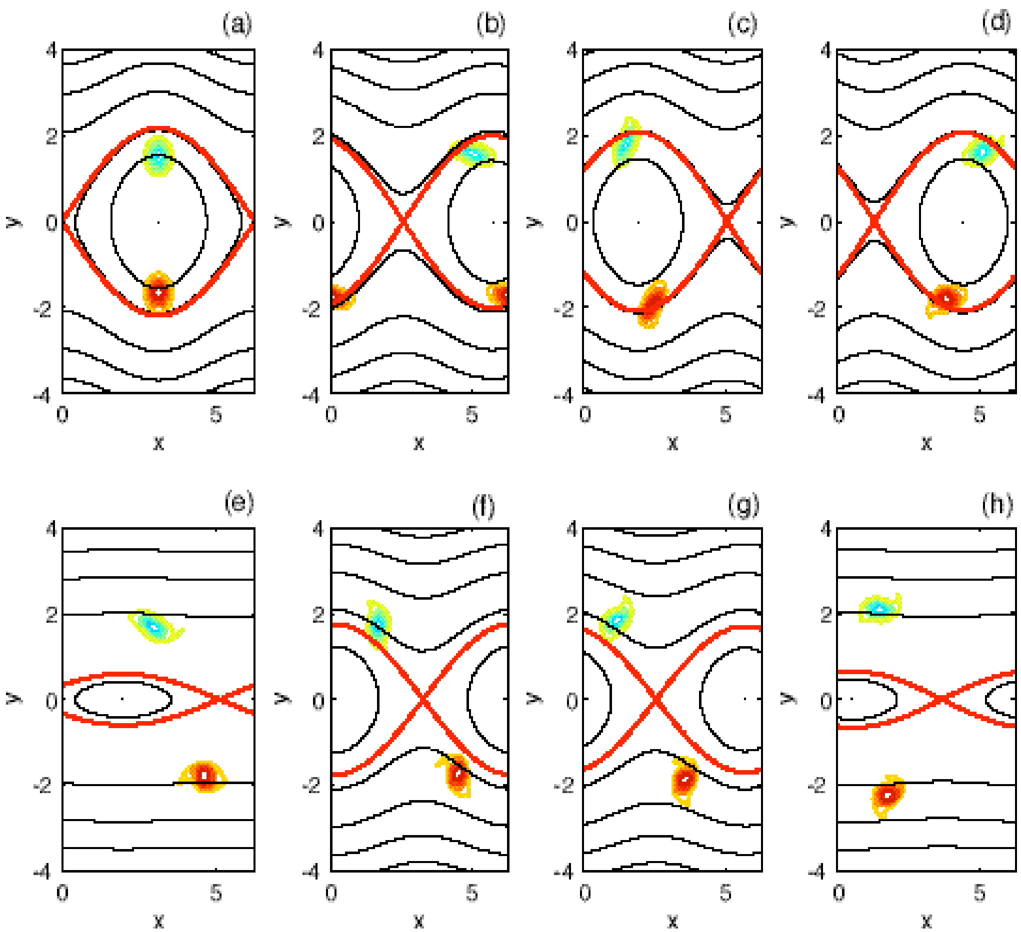
Red: clump
Blue: hole

del-Castillo-Negrete, M.C. Firpo, CHAOS, **12**, 496 (2002).

Asymmetric un-trapped dipole

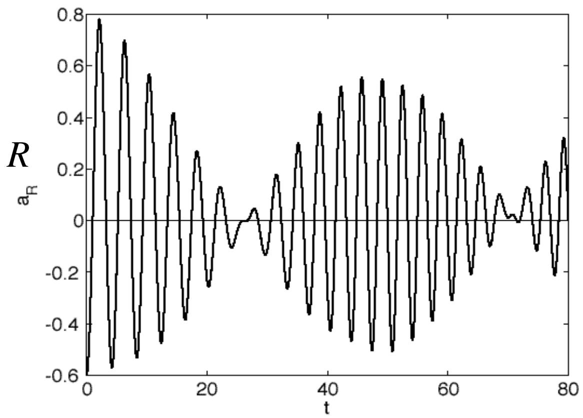
The separatrix drifts and exhibits hyperbolic-elliptic bifurcations

The asymmetry gives rise to quasiperiodic oscillations of the complex mean-field

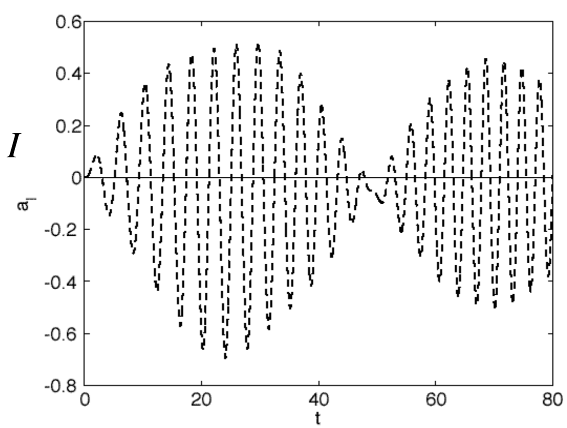


$$a = a_R + i\, a_I$$

a_R



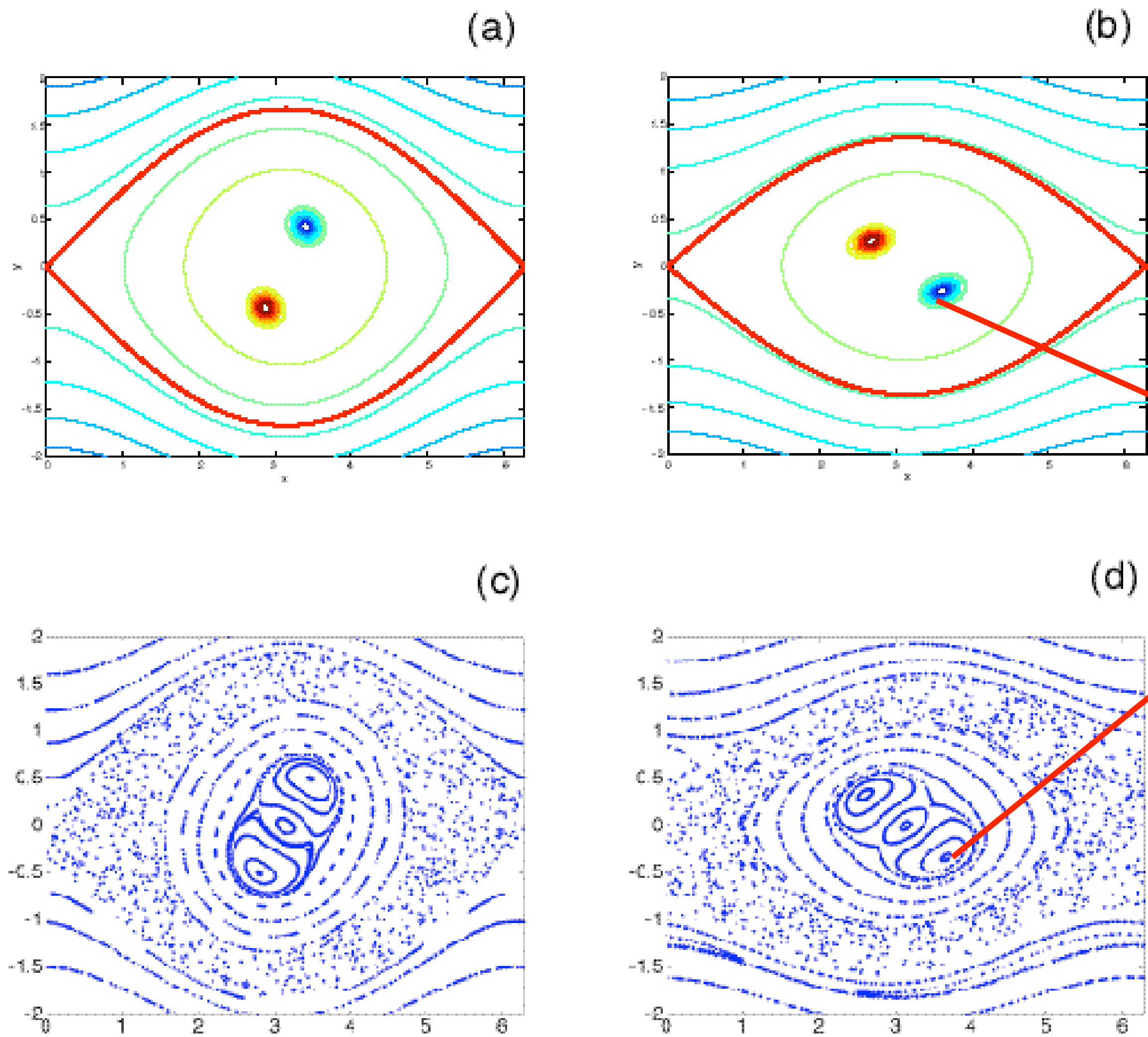
a_I



Self-consistent chaos and coherent structure

Rotating coherent dipole

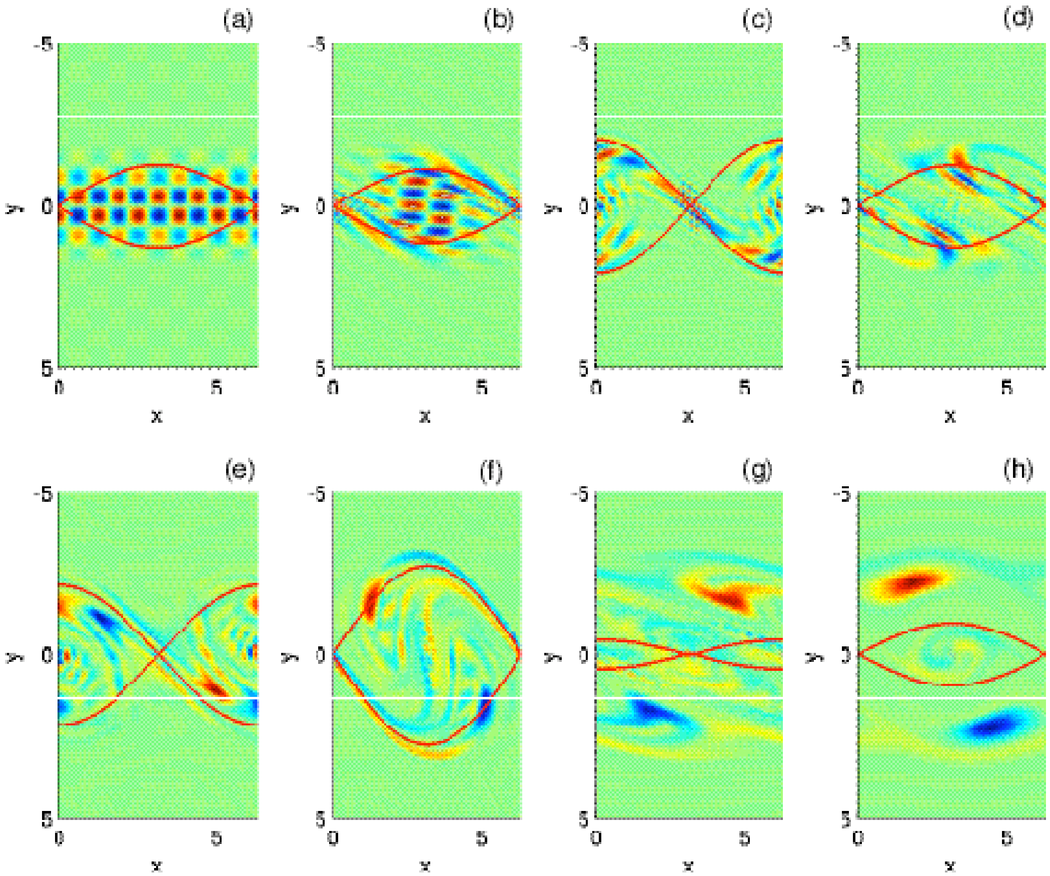
Poincare section of time periodic self-consistent mean-field



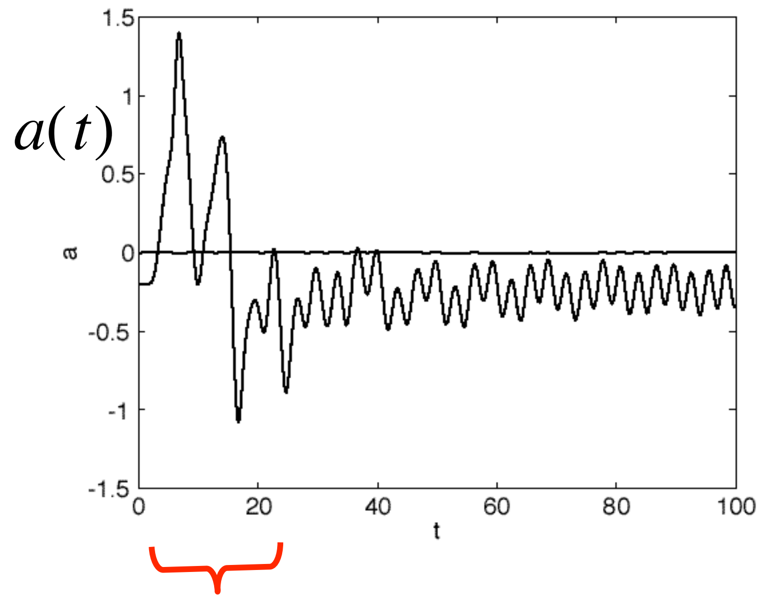
Coherence maintained by KAM surfaces

del-Castillo-Negrete, M.C. Firpo, CHAOS, **12**, 496 (2002).

Hyperbolic-elliptic bifurcations play a key role in the rapid mixing and relaxation of far from equilibrium initial conditions



Self-consistent evolution of wave mean field



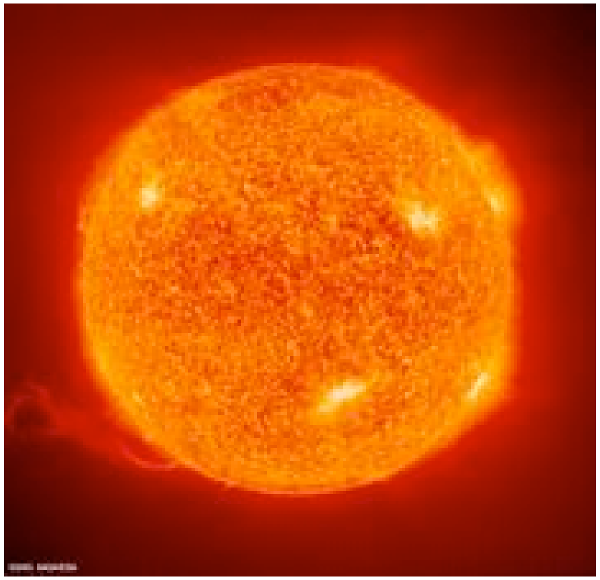
Rapid mixing phase

Del-Castillo-Negrete, Plasma Physics and Controlled Fusion, **47**, 1-11 (2005).

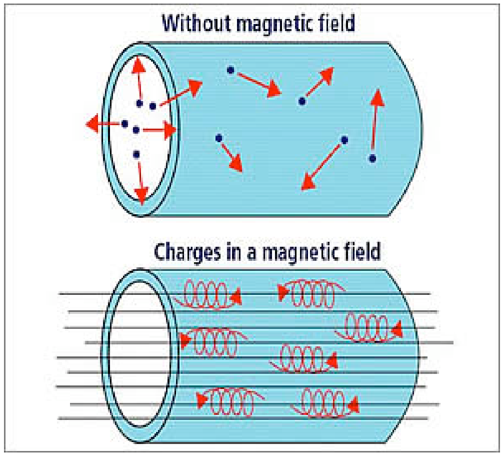
Levy flights and fractional diffusion models of transport

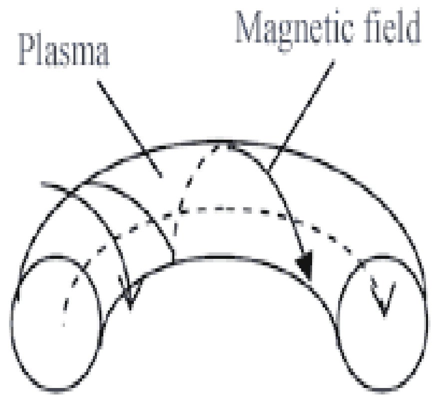
Fusion plasmas

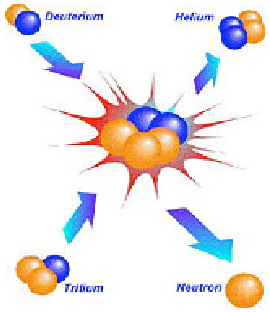
Fusion in the sun

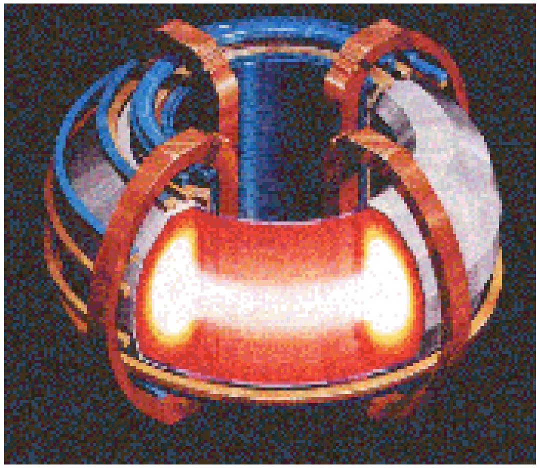


Magnetic confinement









Controlled fusion on earth

- Understanding radial transport is one of the key issues in controlled fusion research

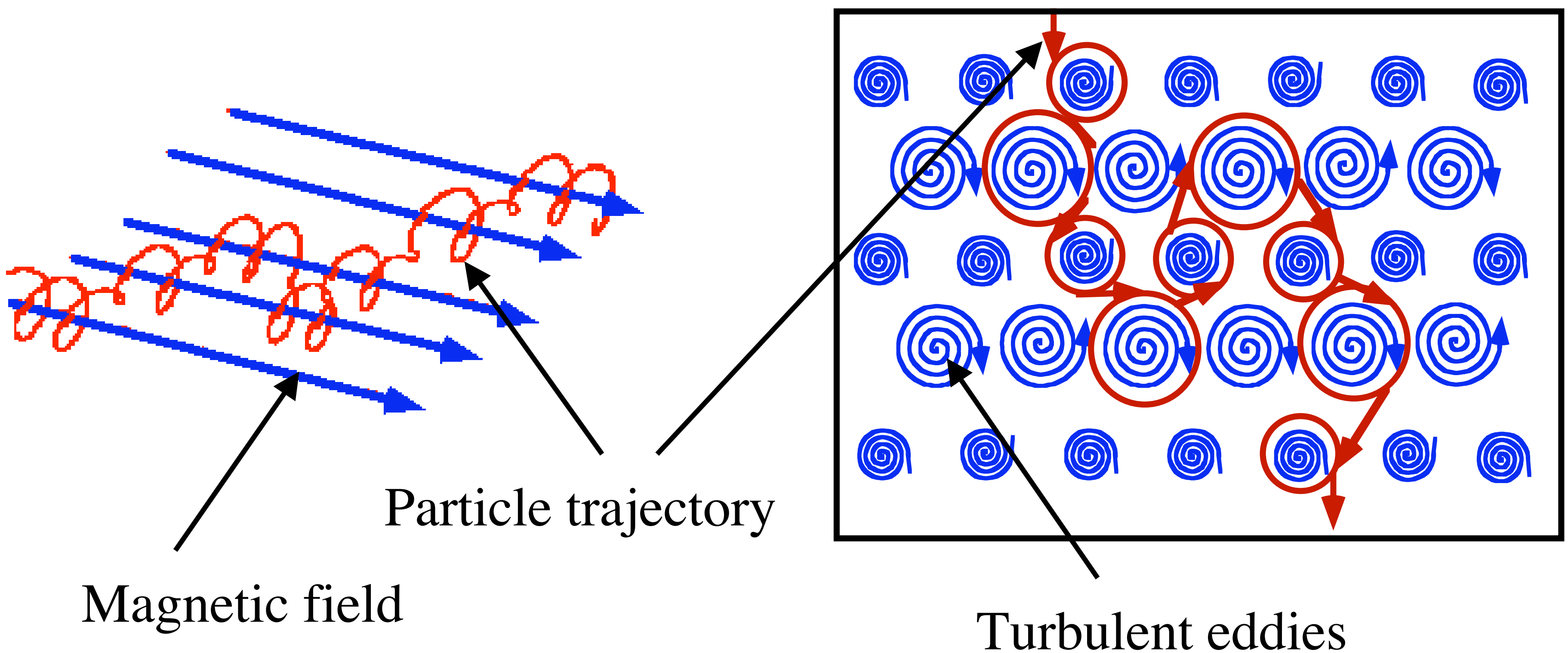
•This is highly non-trivial problem!

•Standard approaches typically underestimate the value of the transport coefficients due to the presence of anomalous diffusion

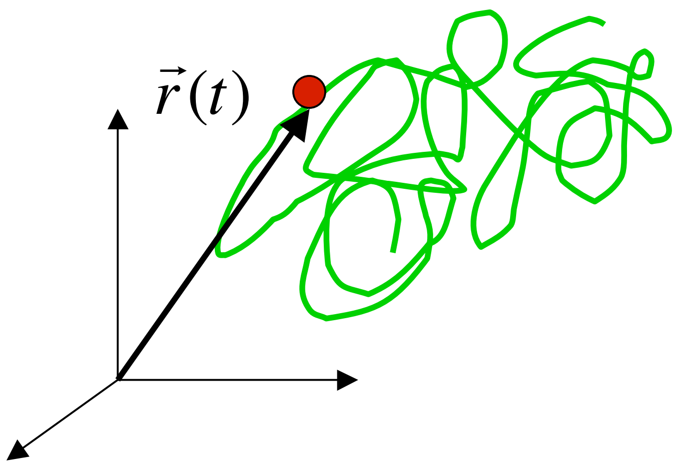
Transport in magnetically confined plasmas

Collisional transport across magnetic field

Turbulent transport



Turbulent transport



$$\delta\vec{r}(t) = \vec{r}(t) - \vec{r}(0)$$

$\langle\ \rangle$ = ensemble average

$$M(t) = \langle \delta\vec{r} \rangle = \text{mean}$$

$$\sigma^2(t) = \left\langle \left[\delta\vec{r} - \langle \delta\vec{r} \rangle \right]^2 \right\rangle = \text{variance}$$

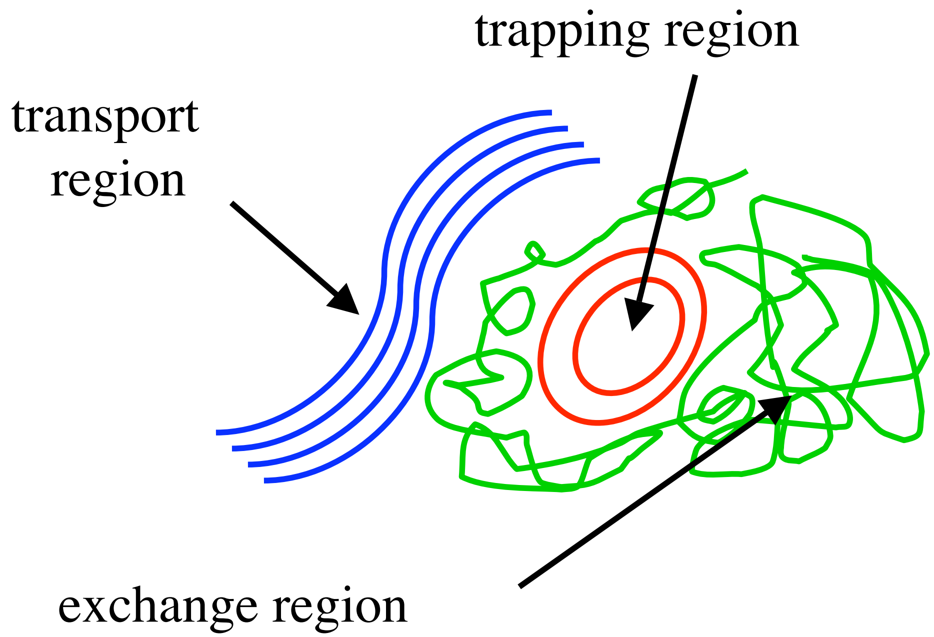
$\mathrm{P}(\delta\vec{r}, t)$ = probability distribution

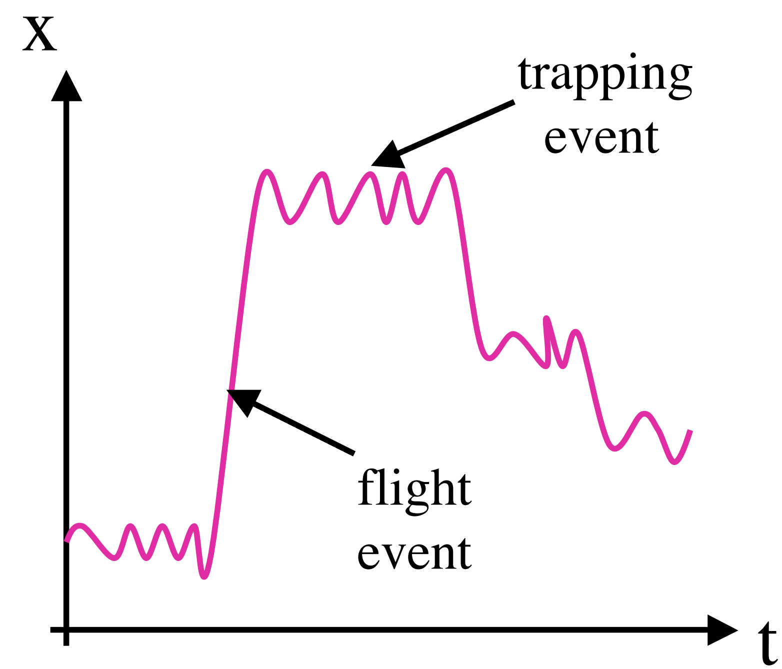
homogenous, isotropic turbulence → Brownian random walk → $\lim_{t \to \infty}$

$$\begin{cases} M(t) = V\,t \\ \sigma^2(t) = D\,t \\ \mathrm{P}(\delta\vec{r}, t) = \text{Gaussian} \end{cases}$$

V= transport velocity   D=diffusion coefficient

Coherent structures can give rise to anomalous diffusion

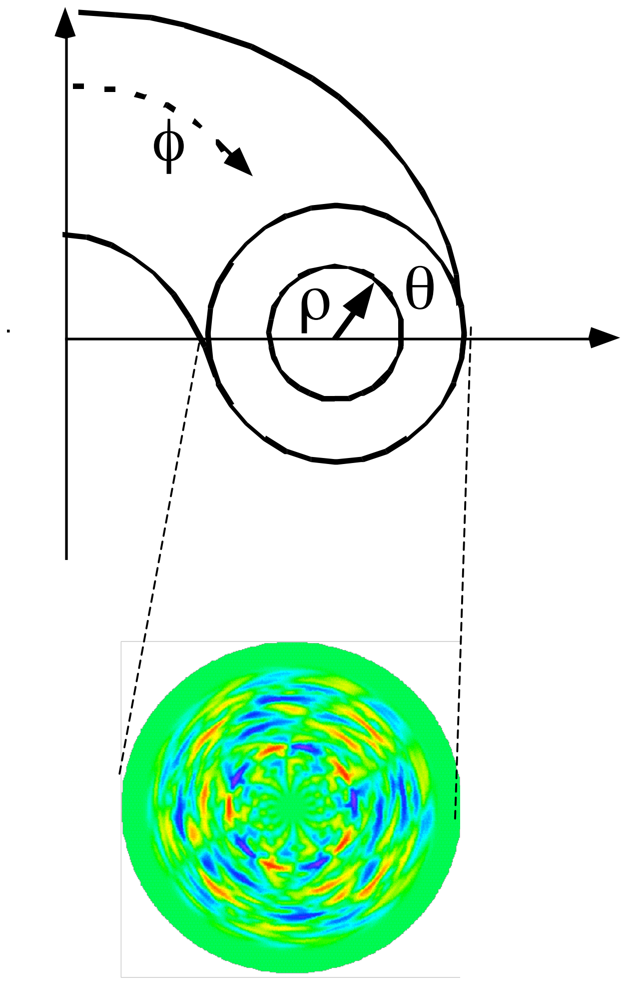




Coherent structures → correlations → $\lim_{t \to \infty} \begin{cases} \sigma^2(t) \sim t^\gamma \\ P(x,t) = \text{non - Gaussian} \end{cases}$

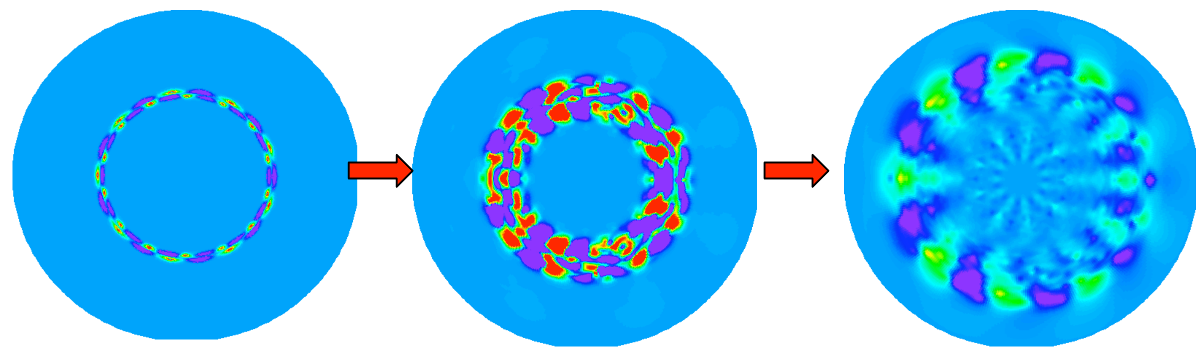
D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

Coherent structures in plasma turbulence

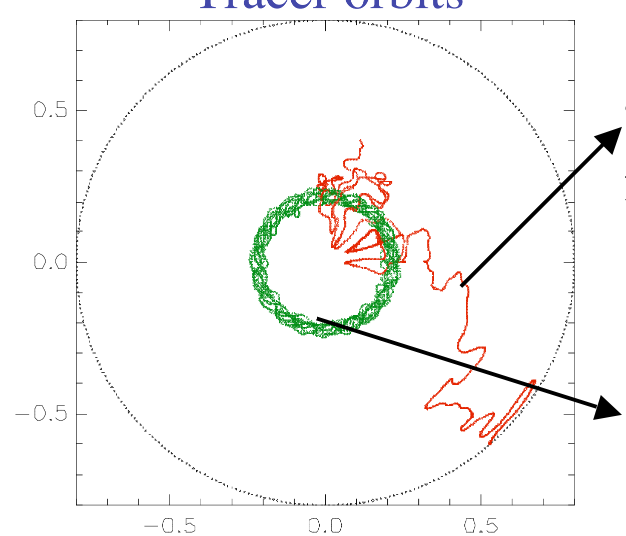


Ex***B*** flow velocity eddies induce particle trapping

"Avalanche like" phenomena induce large particle displacements that lead to spatial non-locality



Tracer orbits



"Levy" flight

Trapped orbit

Combination of particle trapping and flights leads to anomalous diffusion

Anomalous transport in plasma turbulence

3-D turbulence model

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\nabla_\perp^2\tilde{\Phi} = \frac{B_0}{m_i n_0 r_c}\frac{1}{r}\frac{\partial\tilde{p}}{\partial\theta} - \frac{1}{\eta m_i n_0 R_0}\nabla_\parallel^2\tilde{\Phi} + \mu\nabla_\perp^4\tilde{\Phi}$$

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\tilde{p} = \frac{\partial\langle p\rangle}{\partial r}\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\theta} + \chi_\perp\nabla_\perp^2\tilde{p} + \chi_\parallel\nabla_\parallel^2\tilde{p}$$

$$\frac{\partial\langle p\rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}r\left\langle\tilde{V}_r\,\tilde{p}\right\rangle = S_0 + D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle p\rangle}{\partial r}\right)$$

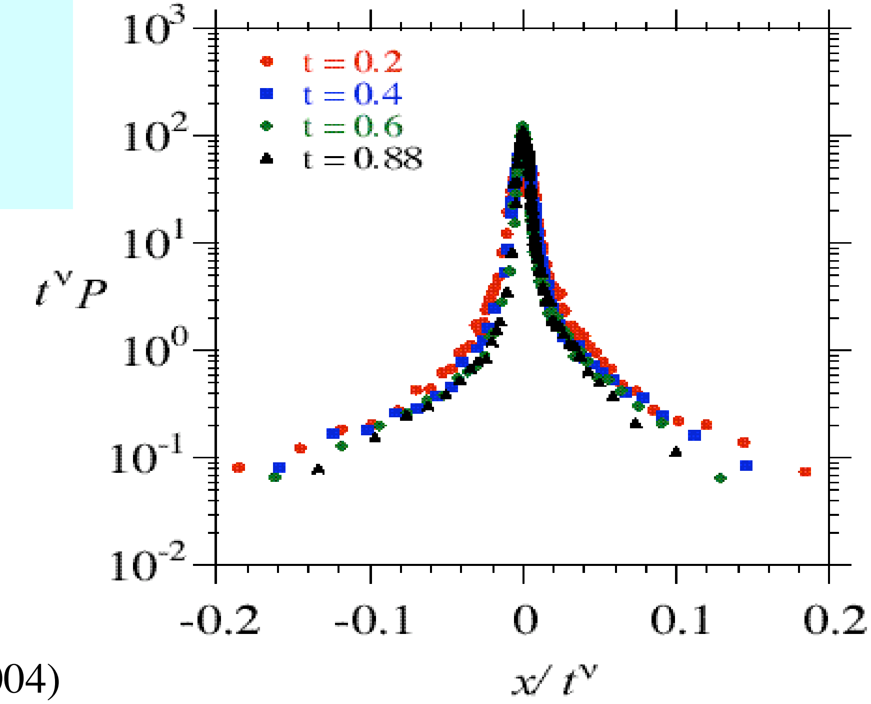
Tracers dynamics

$$\frac{d\vec{r}}{dt} = \vec{V} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

Super-diffusive scaling

$$\left\langle\delta r^2\right\rangle \sim t^{4/3}$$

Levy distribution of tracers displacements

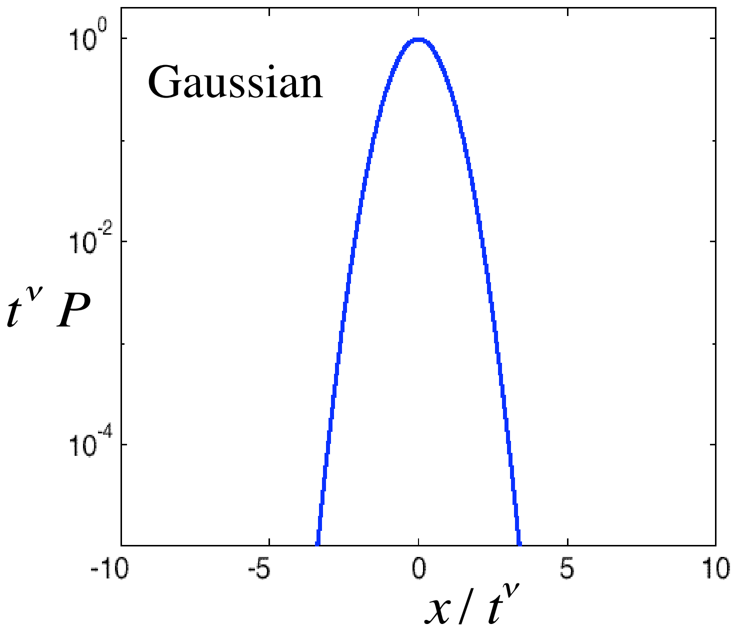


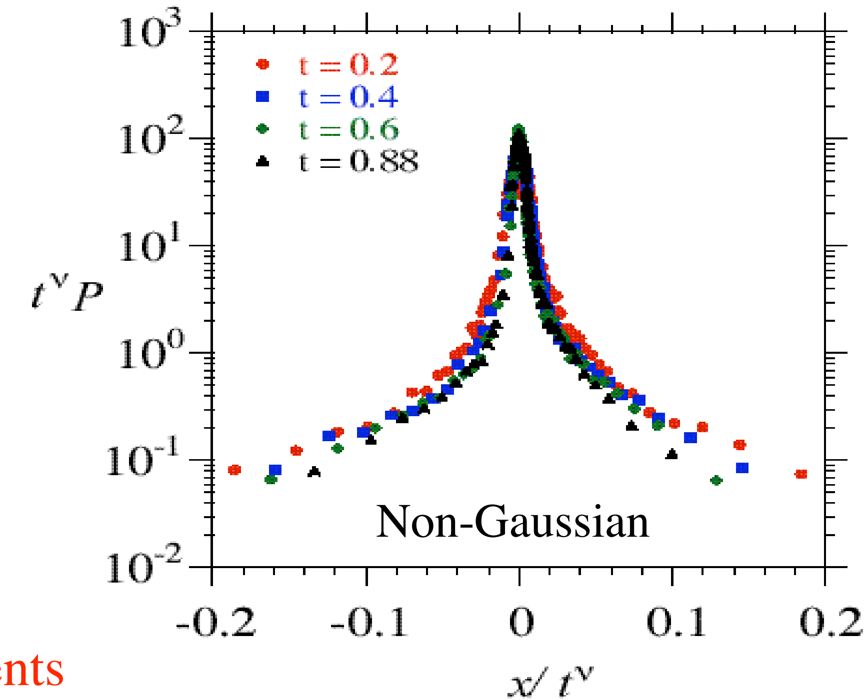
D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004)

Standard diffusion

Plasma turbulence

Probability density function





Moments

Diffusive scaling $\nu = 1/2$

$$\langle x^n \rangle \sim t^{n\nu}$$

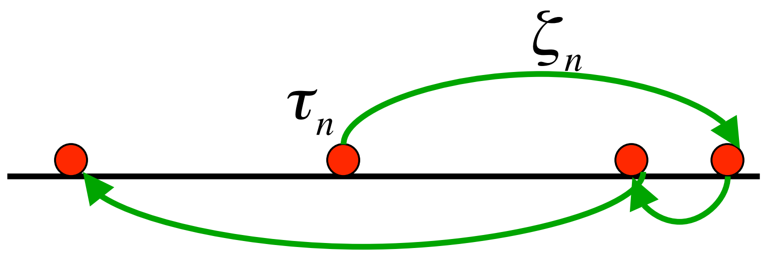
Anomalous scaling super-diffusion $\nu \sim 2/3$

$$\partial_t P = \partial_x \left[\chi \partial_x P \right]$$

Model

??????????

Continuous time random walk model



τ_n = waiting time $\qquad \psi(\tau)$ = waiting time pdf

ζ_n = jump $\qquad \lambda(\zeta)$ = jump size pdf

Master Equation (Montroll-Weiss)

$$\partial_t P = \int_0^t dt'\, \phi(t-t') \int_{-\infty}^{\infty} dx' \left[\lambda(x-x') P(x',t) - \lambda(x-x') P(x,t) \right]$$

$$\tilde{\phi}(s) = s\tilde{\psi}/(1-\tilde{\psi})$$

No memory $\quad \psi(\tau) \sim e^{-\mu\tau}$

Gaussian displacements $\quad \lambda(\zeta) \sim e^{-\zeta^2/2\sigma}$

$$\left.\begin{matrix} \psi(\tau) \sim e^{-\mu\tau} \\ \lambda(\zeta) \sim e^{-\zeta^2/2\sigma} \end{matrix}\right\} \quad \partial_t P = \chi \partial_x^2 P$$

Standard diffusion

Long waiting times $\quad \psi(\tau) \sim \tau^{-(\beta+1)}$

Long displacements (Levy flights) $\quad \lambda(\zeta) \sim \zeta^{-(\alpha+1)}$

$$\left.\begin{matrix} \psi(\tau) \sim \tau^{-(\beta+1)} \\ \lambda(\zeta) \sim \zeta^{-(\alpha+1)} \end{matrix}\right\} \quad {}_0^cD_t^{\beta}\, \phi = \chi\, D_{|x|}^{\alpha}\, \phi$$

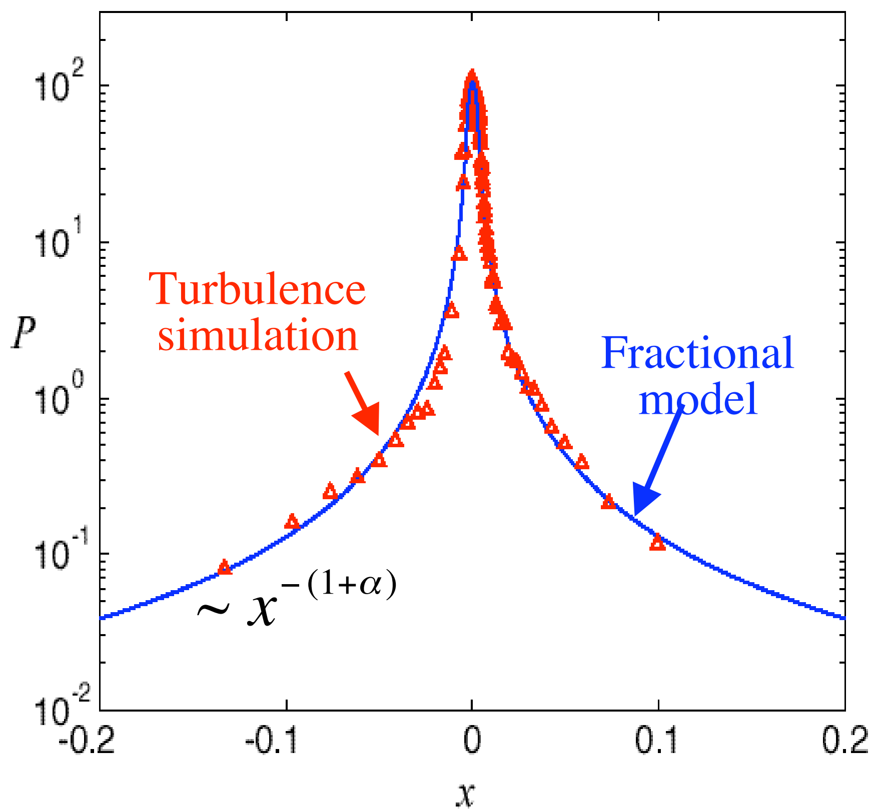
Fractional diffusion

Comparison between fractional model and turbulent transport data

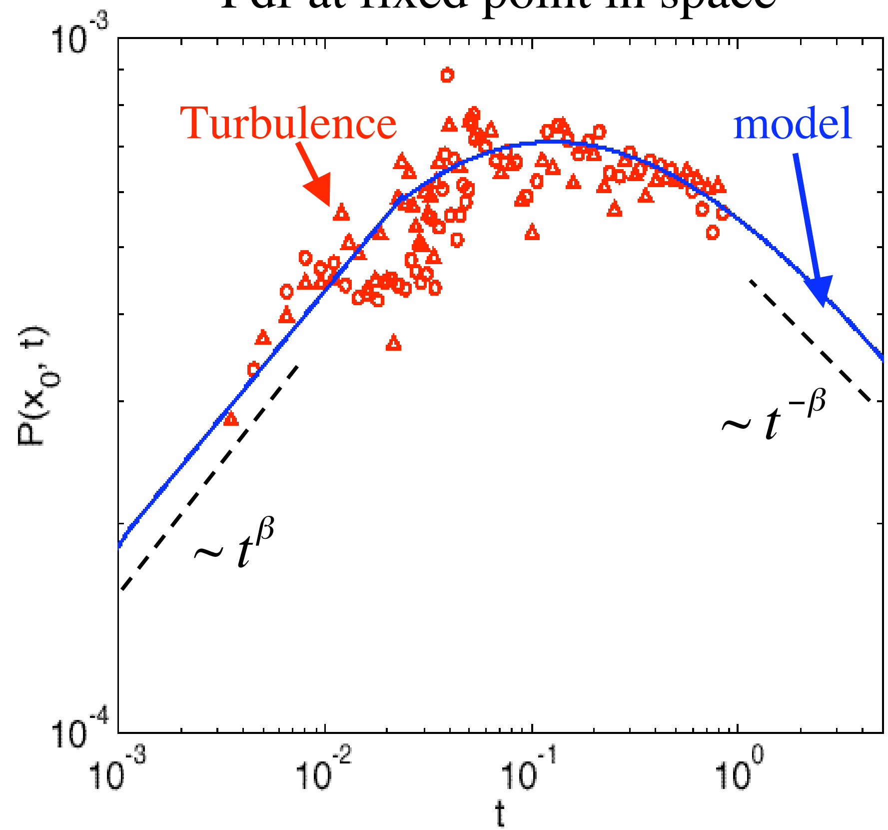
$$\boxed{\alpha = 3/4 \quad \beta = 1/2}$$

$$\langle x^2 \rangle \sim t^{2\beta/\alpha} \sim t^{4/3}$$

Levy distribution at fixed time



Pdf at fixed point in space



D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005); Phys. Plasmas **13**, (2006);