Leaking dynamical systems: a fresh view on Poincaré recurrences

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Two types of transient chaos

I. Open dynamics, the system has passed an
external crisis

Ott, 1993

II. Closed dynamics, the system is opened up (leaked)
artificially

‘Picture an energy conserving billiard table … . Now suppose a small hole is cut in the table so that the ball can fall through.’

Pianigiani, Yorke, 1979

Leaked billiards

Chaotic billiard with a small hole of size Δ along the boundary.

Exponential decay in the probability to escape at time n

on a Poincaré map

$$p_e\,(n) \sim \exp(-\gamma_e\, n)$$

with escape rate

$$\gamma_e = \Delta/P = \mu(\text{hole}) = 1/\langle n\rangle_e,$$

P: perimeter of billiard
μ: natural measure

for $\mu(\text{hole}) \ll 1$.

Bauer, Bertsch, 1990

3D billiards in real time

$$\langle t\rangle_e = 4\,V/(c\,A),$$

V: volume of billiard
A: area of leak; c: velocity

Jaeger, 1911

Sabine's law

Wallace C. Sabine (1868-1919),
founder of architectural acoustics:

Residual sound intensity decays exponentially in time.

Reverberation time T: duration to decay below the audible intensity
(a factor of 10^{-6}, 60dB)

1898: **T=0.16 V/A** (SI units)

V: room volume, A absorbing area (or equivalent area)

Independent of the location of source,
of details of the room,
provided the room is 'well mixing',
review: Mortesagne, Legrand, Sornette, 1993

Sabine's law

Note that:

$T=6\ \ln 10 \langle t \rangle_e = 6\ \ln 10\ 4\ V/(c\ A) = 0.16\ V/A$,

with c=sound velocity.

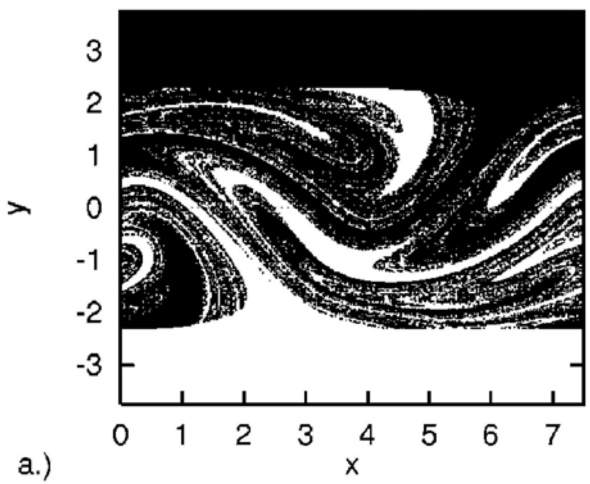
Sabine's law seems to be the first application of the concept of transient chaos (energy is escaping) and leaking (via energy-absorbing surfaces).

Resetting mechanism

Simple model of coloring fluid elements according to their history (a reaction model)

Pierrehumbert, 1994

Two different dyes maintained at the saturation concentration in the boundary layer along two plates in a time-periodic chaotic flow.
Dye I (activated state), Dye II (deactivated state)



Resetting region, Dye I: black

Resetting region, Dye II: white

Neufeld et al, 2000

Resetting is a kind of leaking. The leak is not small :

$\boldsymbol{\gamma_e \neq \mu(\text{leak}), \quad \langle n \rangle_e \neq 1/\gamma_e}$

Invariant sets related to the resetting mechanism

Never resetted points (either forward or backward):

chaotic saddle in the leaked flow

Fractal part of the dye boundary: stable manifold of the saddle in the time reversed dynamics

unstable manifold in the direct dynamics

How do the properties of transient chaos change when changing the resetting region, the leak?

Leak properties: location, size, shape, orientation.

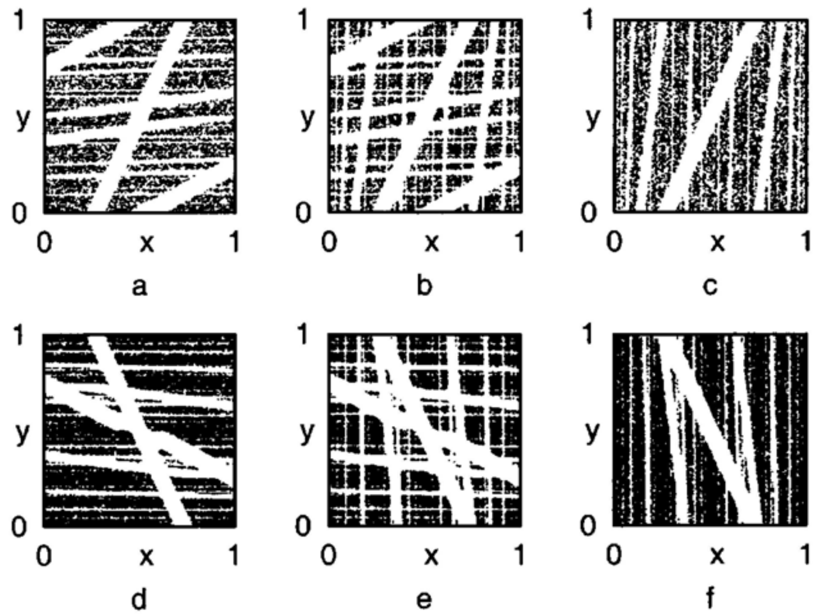
Leaking the baker map

The simplest area-preserving version of baker maps with uniform hyperbolicity: $\lambda = \ln 2$.

Consider a single, but extended leak, a band of area μ, at different tilt angles:

Angle:25°, $\mu=0.1$.

Angle:-25°, $\mu=0.1$.



Schneider et al 2002

stable m. saddle unstable manifold

Constructing the invariant sets

Out of N>>1 trajectories keep those who survive outside of the leak n>>1 steps ($N=10^6$, $n=40$). They come close to the saddle.

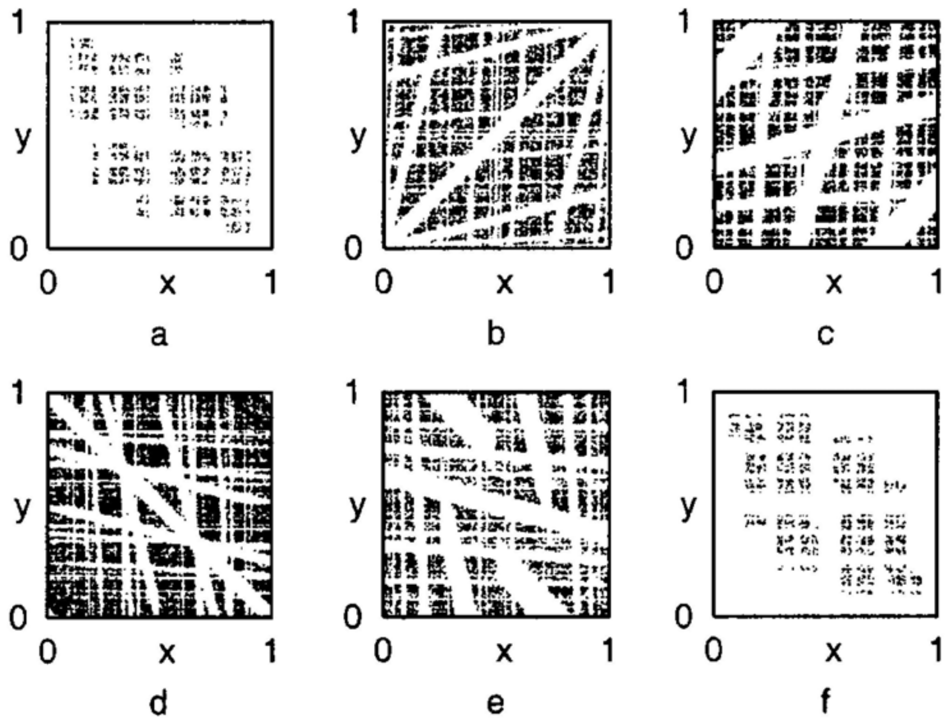
T.T, M. Gruiz, 2006

Initial points (i=0) of these trajectories: on the **stable** manifold

End points (i=n): on the **unstable** manifold

Midpoints (i=n/2): on the chaotic **saddle**.

Chaotic saddles at different tilt angles

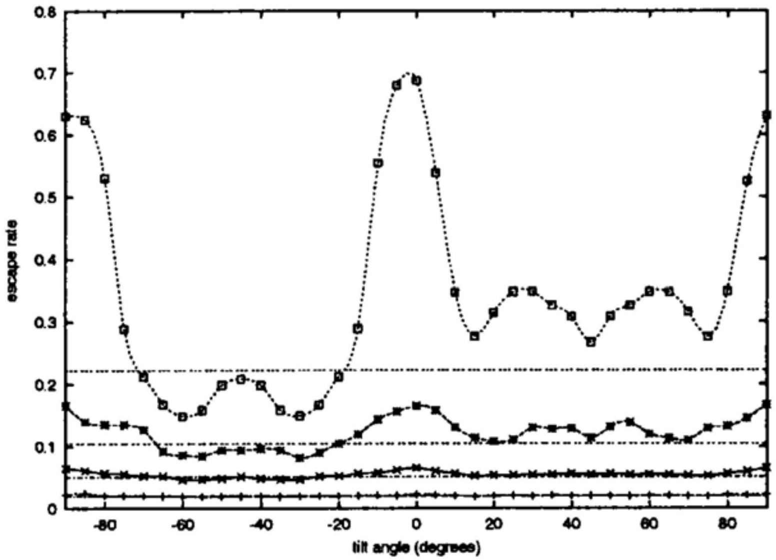


Schneider et al, 2002

Tilt angles: 0, 45°, 75°, -45°, -15°, -90°, μ=0.1.

The fractal dimension and the escape rate is angle-dependent.
Note: **λ(saddle)=ln2** in all cases.

Dependence of the escape rate on the orientation of the leak



μ=0.2

μ=0.1

Naiv estimate: **exp(- γ_e)= (1-μ), γ_e = - ln(1-μ)** horizontal line.
Only valid for μ<<1, when **γ_e =μ**.

Topological entropy: **h= λ(saddle)- γ_e =ln2- γ_e** Kantz,Grassberger, 1985

Different leaks produce drastically different pruning of the symbolic dynamics.

Mixing properties in the Earth's mantle, via leaking

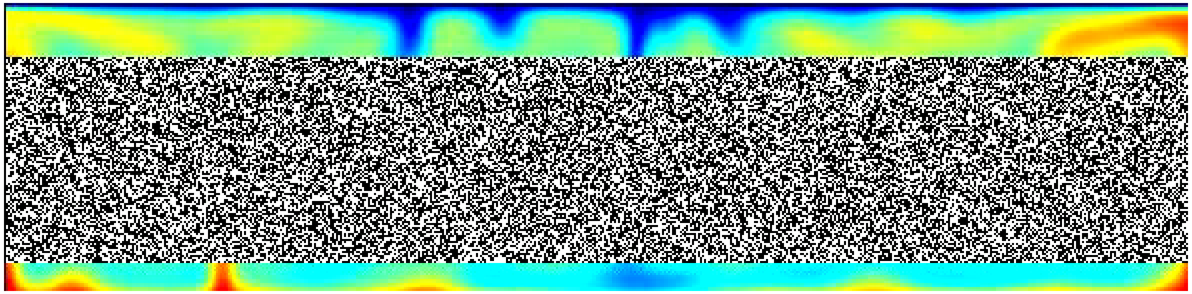
Thermal convection simulated in a 2D rectangular domain of aspect ratio 1:4.
Schneider, Schmalzl, T.T, 2007

Rayleigh number: 10^7, Prandtl number : infinite

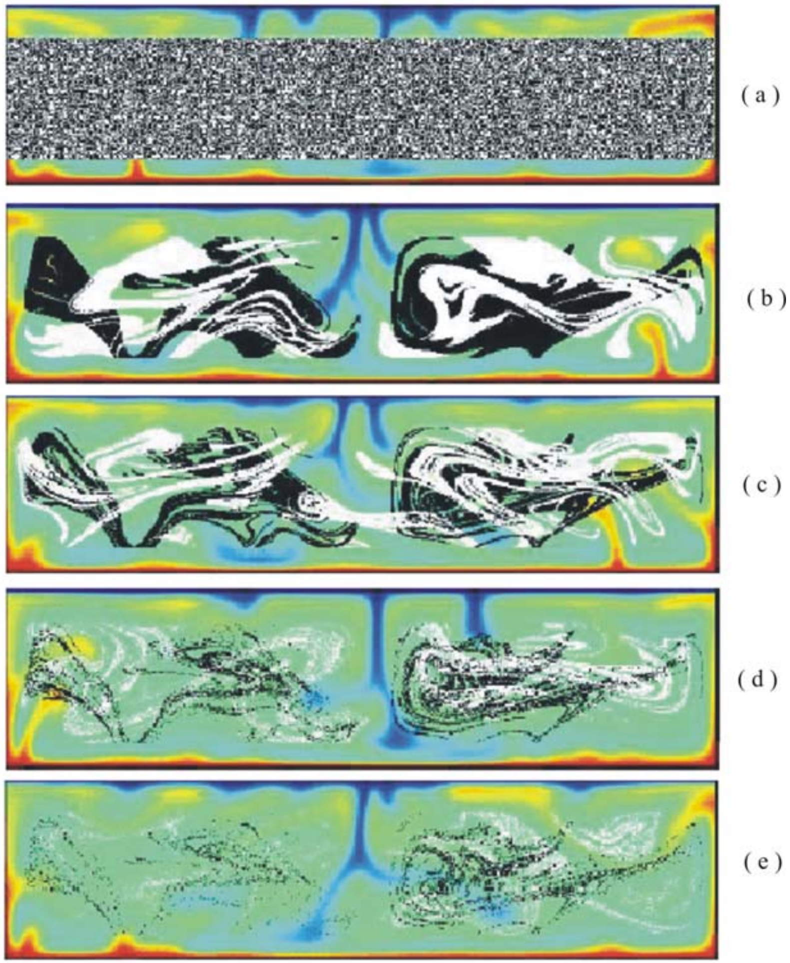
↓

Flow: irregular, but cellular pattern on average.
Flow with chaotic time-dependence.

Colour coding: temperature, leak: complement of the dotted region



Time evolution of the manifolds



Time: 2 overturns

Time: 6 overturns

Black: stable manifold
White: unstable manifold

Time: 14 overturns

Time: 17 overturns

Explanation of clear fractality: Theory of random maps, snapshot attractors

Romeiras, Gebogi, Ott, 1990

Material exchange

Inhomogeneities remain after 10 overturns.

1 overturn: 400-500 million years! At present, the Earth mantle is inhomogeneous.

The lifetime of Earth ($5\ 10^9$ years: 10 overturns(!)) has not been long enough

to reach a well-mixed state, in spite of chaos.

Other aspects of leaking

Leaked quantum billiards: chaotic spectroscopy

Doron, Smilansky, 1992

Dissipative systems: Leaking chaotic attractors. Relevant for the OGY control of chaos

Paar, Pavin,1997
Buljan, Paar, 2001
Bunimovich, Yurchenko, poster

A recent exact result: two small leaks A and B. The joint escape rate

$$\gamma_e(AB)= \gamma_e(A)+ \gamma_e(B) + \text{sum of correlation functions}$$

The reason: overlaps of the premiages

Bunimovich, Dettman 2007

Poincaré recurrences

Choose a region, I, the recurrence region, in the phase space of a closed map. Start a trajectory in I. There is a finite probability $\mathbf{p_r(n)}$ for returning to I at some finite time instant n. $\langle n \rangle_r$= finite.

Henri Poincaré (1854-1912)

Poincaré recurrences proved to be useful analysers of low-dimensional chaotic systems.

Chirikov, Shepalyansky, 1984

Known facts:

Recurrence probability

$\mathbf{p_r\ (n) \sim exp(-\gamma_r\ n),}$ γ_r : decay exponent .

The average recurrence time

$\mathbf{< n >_r = 1/\ \mu(I)}$ Kac's lemma 1959

valid for dissipative systems, too

$\mathbf{\gamma_r \neq 1/< n >_r = \mu(I)}$

The decay exponent cannot be related to **μ(I),**

unless μ(I)<<1.

Recurrence vs leaking

Consider a leaked system with the leak being identical with the recurrence region I

Altmann, T.T. 2007, 2008

Compare the recurrence and escape distributions:

For $n>n^*_r$

$\mathbf{p_r (n) = g_r \exp(-\gamma_r\, n)}$

$\mathbf{< n >_r = 1/ \mu(I)}$

For $n>n^*_e$

$\mathbf{p_e (n) = g_e \exp(-\gamma_e\, n)}$

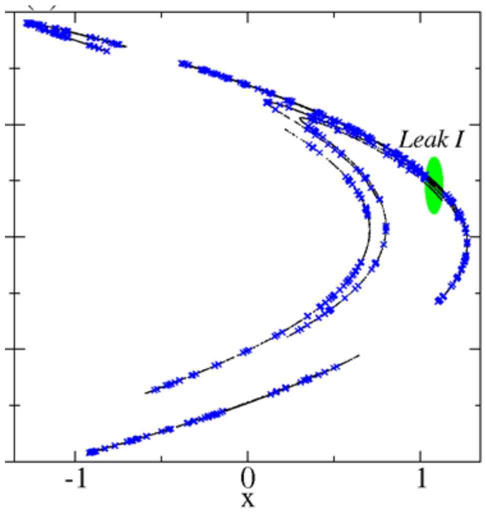
$\mathbf{< n >_e, n^*_e , g_e}$
also depend on the initial distribution ρ_0

$$\mathbf{n^*_r \neq n^*_e ,\quad < n >_r \neq < n >_e ,\quad g_r \neq g_e}$$

The underlying chaotic saddle

Chaotic saddle of the leaked Hénon attractor (a=1.4, b=0.3);

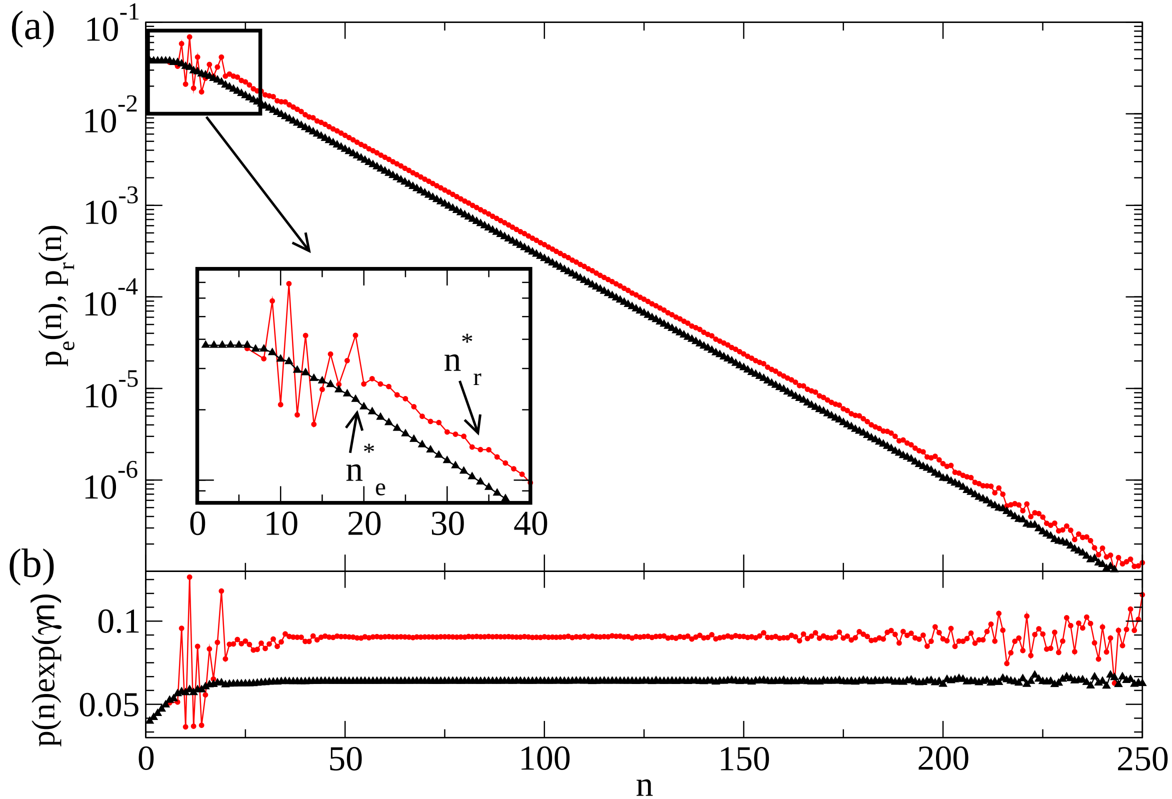
the leak I is a circle of radius 0.05.



The same saddle must be visited by trajectories having long recurrence times:

$$\gamma_r = \gamma_e$$

Numerical evidence



Red: recurrence, $\gamma = 0.055$

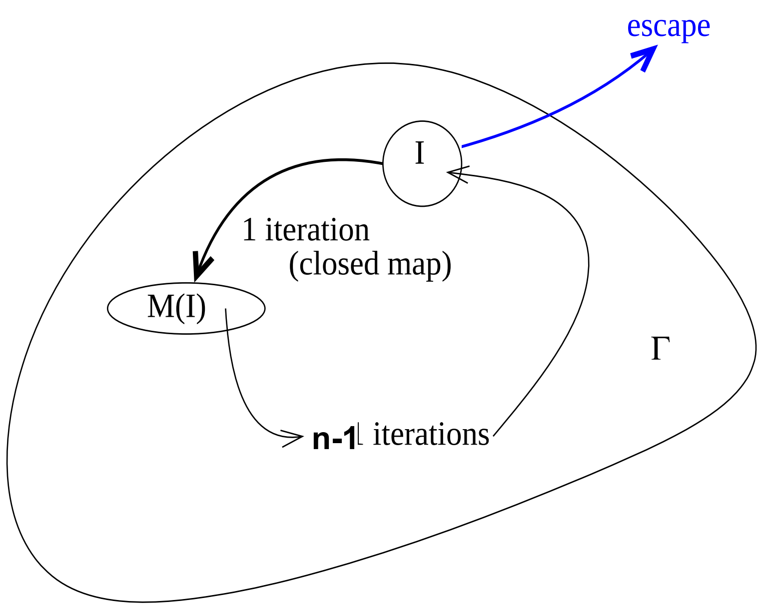
black: escape, ρ_0: natural distr. on the Hénon attractor

Special initial condition for full equivalence

Consider the density ρ_I of the natural measure μ inside the leak, I.

Take its first iterate $M(\rho_I)$ with map M as the initial density for the

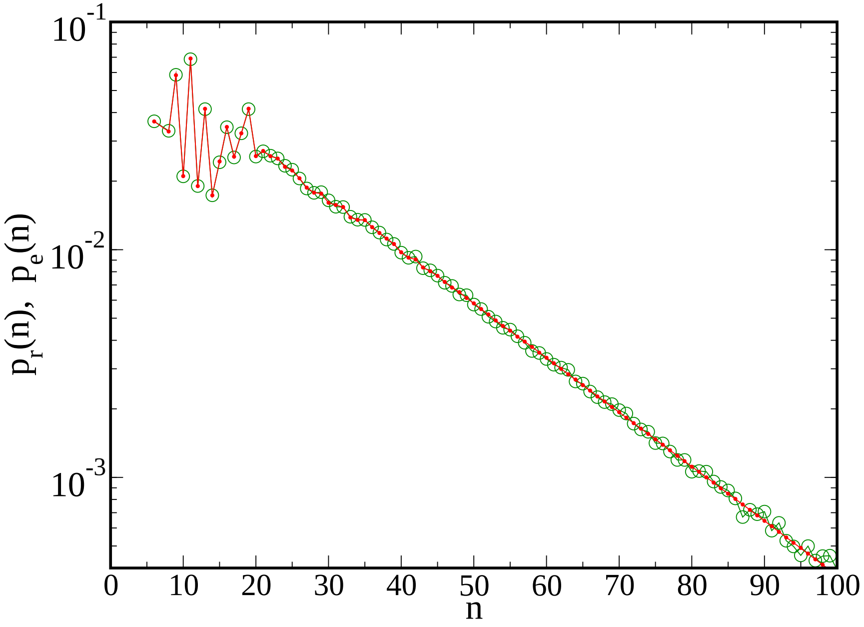
escape dynamics of the leaked system: $\rho_0 = M(\rho_I)$.



With this ρ_0 the two statistics coincide: **$p_r(n) = p_e(n)$.**

Special initial condition for full equivalence

Numerical evidence:



Red dots: recurrence, green circles: escape with $\rho_0 = M(\rho_I)$

Conditionally invariant measure for the leaked problem

A central concept in the theory of transient chaos: natural distribution of the orbits which have not yet escaped: $\boldsymbol{\mu_c}$

Pianigiani, Yorke, 1979

It is concentrated along the unstable manifold of the chaotic saddle.

In general $\mathbf{exp(-\gamma_e)=\mu_c(nonescaping\ region\ in\ one\ step)}$

$\boldsymbol{\mu_c(\Gamma \backslash I) = \mu_c(\Gamma) - \mu_c(I) = 1 - \mu_c(I)}$

Γ: phase space

$\boldsymbol{\gamma_e = -\ln(1 - \mu_c(I))}$

The c-measure might depend strongly on the location, even if $\boldsymbol{\mu(I)}$ does not.

As a consequence:

$\boldsymbol{\gamma_r = -\ln(1 - \mu_c(I))}$

the decay rate of the recurrence distribution can be expressed by the c-measure of I.

Hamiltonian systems with KAM tori

For recurrence/leak region far from KAM tori,

the intermediate-time decay is exponential,

but the asymptotic long-time decay is a power-law:

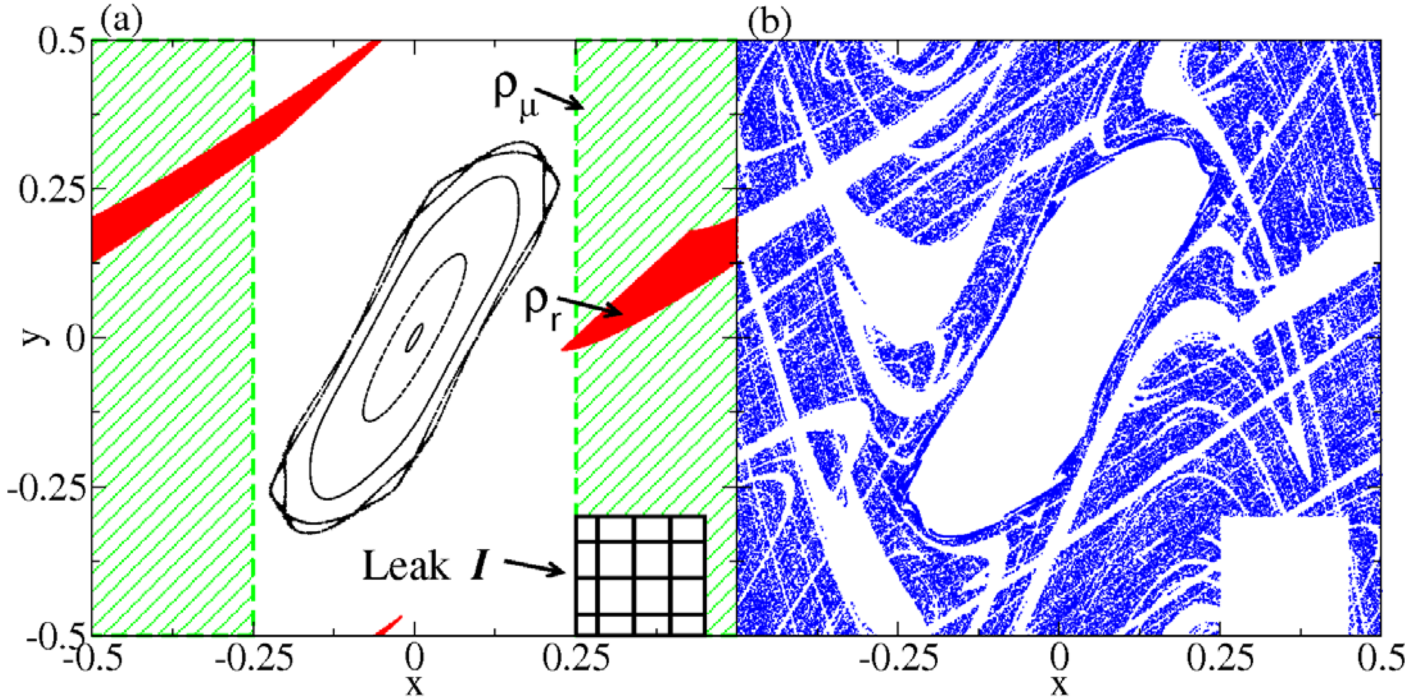
$$p_{r,e}(n) \sim \exp(-\gamma_{r,e}\, n) \qquad n^* < n < n_c$$

$$\sim 1/n^{\sigma_{r,e}} \qquad n_c < n$$

There is an underlying nonhyperbolic chaotic saddle

Hamiltonian systems with KAM tori

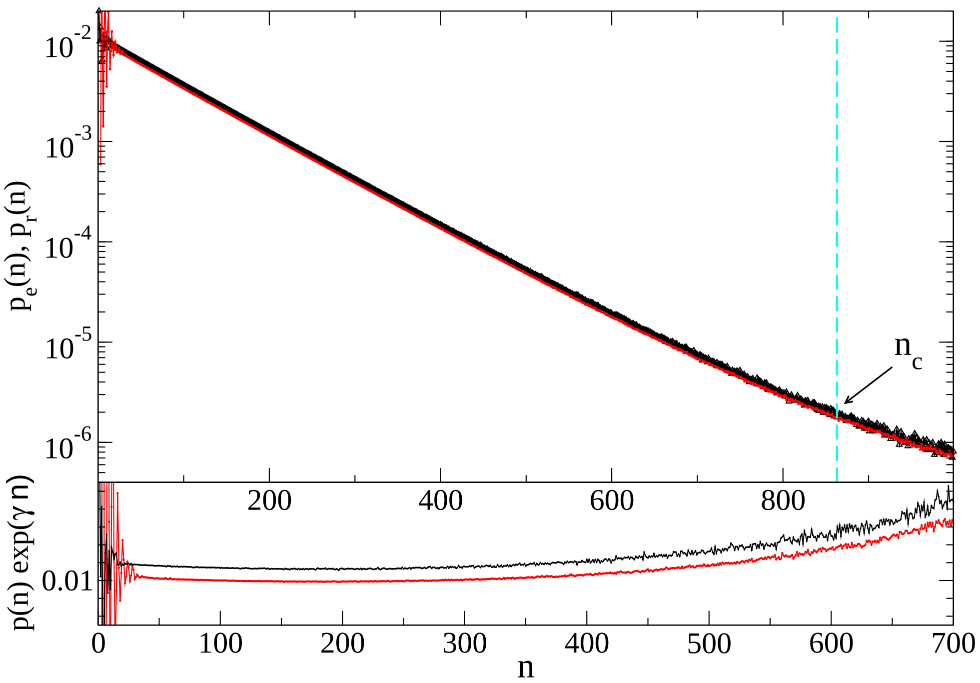
The chaotic saddle also contains the KAM torus, standard map, K=0.52



Since the saddle is the same for both escape and recurrence:

$$\gamma_r = \gamma_e, \qquad \sigma_r = \sigma_e$$

Numerical evidence:



Red: recurrence, $\gamma=0.01$

black: escape, ρ_0 = constant outside of the torus (green region)

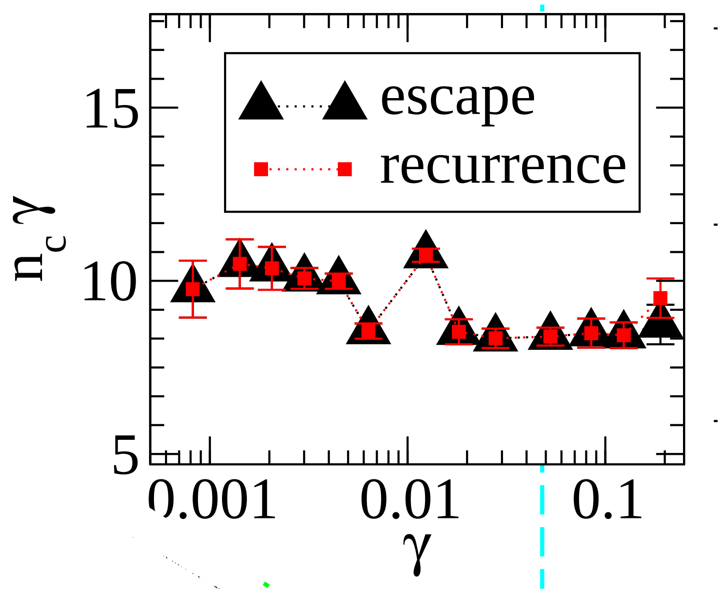
Full equivalence with the red initial condition, $\rho_0 = M(\rho_I)$

Numerical evidence

The crossover time n_c for which $\exp(-\gamma n_c)=n_c^{-\sigma}$ scales as:

Altmann, T.T., 2007, 2008

$n_c \sim 1/\gamma$



For small γ, the exponential decay is very long.
It is meaningful to split the saddle into a hyperbolic and a nonhyperbolic component. The hyperbolic one is visited first, the nonhyperbolic one afterwards.

Summary

An unified perspective for leaked systems and Poincaré recurrences is obtained, which has not been available earlier, without the use of leaked dynamics. The description is given in terms of a chaotic saddle, the decay exponent can be written in terms of the conditionally invariant measure.

Transient chaos generated by leaking dynamical systems is a useful analyser of permanent chaos.

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