Stability and Feedback Control of Frictional Dynamics for A One-Dimensional Particle Array

Yi Guo
Department of Electrical and Computer Engineering
Stevens Institute of Technology
Email: yguo1@stevens.edu

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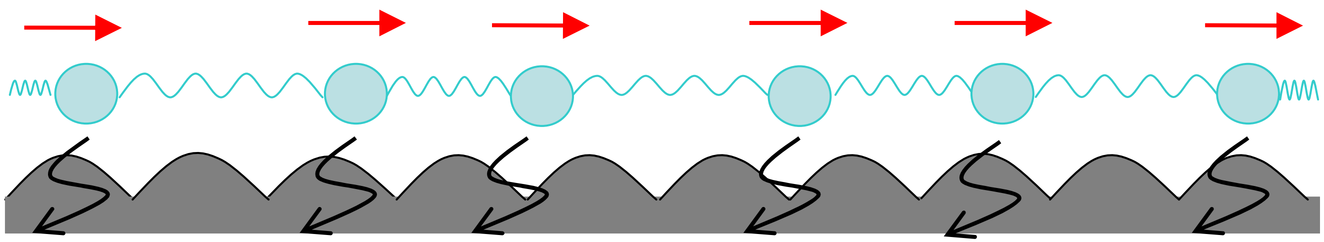
Outline

- Motivation
- The Model and Control Problem Formulation
- Open-Loop Stability
- Tracking Control Design
- Single Particle Dynamics
- Conclusion

Motivation

- Control of friction during sliding is important for a variety of applications
- Friction can be manipulated by applying small perturbations to accessible elements and parameters of the sliding system
- [Braiman PRL2003] presents a feedback control scheme to control friction at the nanoscale
- We follow the line of the research to study Lyapunov stability and design precise control of a one-dimensional particle array represented by the Frenkel-Kontorova Model

The Frenkel-Kontorova Model



A harmonic chain (mimic a layer of nano-particles) in a spatially periodic potential (mimic the substrate), driven by a constant force which is damped by a velocity-proportional damping.

The FK-Model

- Dynamics of a one dimensional particle array moving on a surface:

$$m\ddot{x}_i + \gamma \dot{x}_i = -\frac{\partial U(x_i)}{\partial x_i} - \frac{\partial W(x_i - x_j)}{\partial x_i} + f_i + \eta(t)$$

Substrate potential

Inter-particle potential

External force

noise

The FK-Model

- Under simplifications:
  - Sinusoidal substrate potential
  - Zero misfit length between the array and the substrate
  - Same force is applied to each particle
  - Zero noise
- The simplified FK-model:

$$\ddot{\phi}_i + \gamma \dot{\phi}_i + \sin(\phi_i) = f + F_i$$

Dimensionless Phase variable

External force

Particle interaction

The FK-Model

- Morse-type (nonlinear) particle interaction:

$$F_i = \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right\} - \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right\}$$

- As $\beta \to 0$, the linear particle interaction:

$$F_i = \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right)$$

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The FK-Model

- We assume free-end boundary conditions:
  - For the linear case:

$$F_1 = \kappa(\phi_2 - \phi_1), \qquad F_N = \kappa(\phi_{N-1} - \phi_N).$$

  - For the nonlinear case:

$$\begin{aligned} F_1 &= \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_2-\phi_1)} - e^{-2\beta(\phi_2-\phi_1)}\right\}, \\ F_N &= -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_N-\phi_{N-1})} - e^{-2\beta(\phi_N-\phi_{N-1})}\right\}. \end{aligned}$$

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Problem Formulation

- Open-loop stability of the system without external force:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) \;=\; F_i$$

- Tracking control using accessible variables:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) \;=\; F_i + u(t)$$

Design a feedback control law

$$u(t) = u(v_{target}, v_{cm}, \phi_{cm}),$$

where v_{target} is a positive constant, such that v_{cm} tracks v_{target}, and the tracking error tends to zero as t tends to ∞.

Problem Formulation

- Accessible variables (average quantities):

The velocity of the center of mass:

$$v_{cm} = \frac{1}{N}\sum_{i=1}^{N}\dot{\phi}_i,$$

The phase of the center of mass:

$$\phi_{cm} = \frac{1}{N}\sum_{i=1}^{N}\phi_i.$$

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Open-Loop Stability

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) \;=\; F_i$$

- State-space representation:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -\sin x_{i1} - \gamma x_{i2} + F_i \end{aligned}$$

where F_i has two different forms:

– Linear: $F_i \;=\; \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right)$

– Nonlinear: $$\begin{aligned} F_i \;=\; & \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right\} \\ & -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right\} \end{aligned}$$

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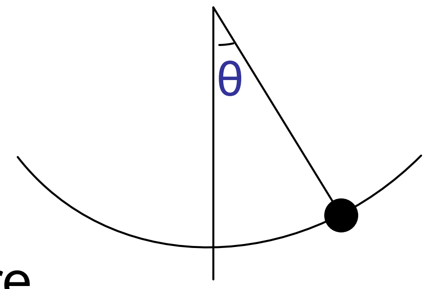
Open-Loop Stability

$$\boxed{\begin{aligned}\dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -\sin x_{i1} - \gamma x_{i2}\end{aligned}} + F_i$$

Pendulum equation



Equilibrium points of the pendulum equations are at kπ

2kπ are stable equilibrium
(2k+1)π are unstable equilibrium (saddle points)

The Equilibrium Points

- For nonlinear system $\dot{x} = f(x)$, the equilibrium points are obtained by $f(x^*)=0$

- In the case of linear particle interaction, the equilibrium is at $(x_{i1},x_{i2})=(x_{i1}^*,0)$,where x_{i1}^* are solutions to

$$
\begin{aligned}
&-\sin x_{11}^* + \kappa(x_{21}^* - x_{11}^*) = 0,\\
&-\sin x_{i1}^* + \kappa(x_{i+1,1}^* - 2x_{i1}^* + x_{i-1,1}^*) = 0,\\
&\qquad i = 2,\ldots,N-1,\\
&-\sin x_{N1}^* + \kappa(x_{N-1,1}^* - x_{N1}^*) = 0
\end{aligned}
$$

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- An example system: N=3, κ=0.26, γ=0.1
- The system may have infinite number of equilibrium points
- Two sets of equilibrium points are at:
  - Set 1:

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.19, 0, 0.93, 0, 4.77, 0)$$

  - Set 2:

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.69, 0, 3.14, 0, 5.59, 0)$$

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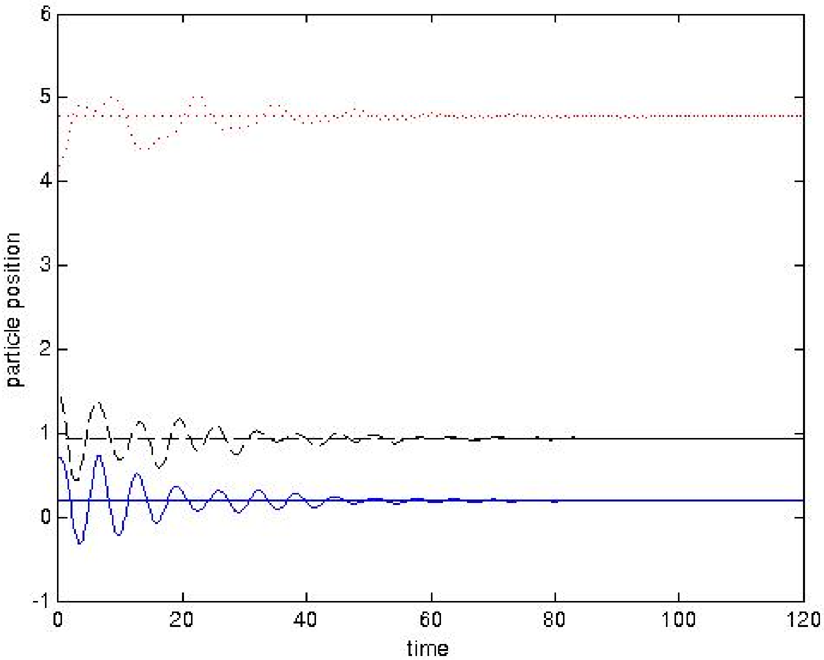
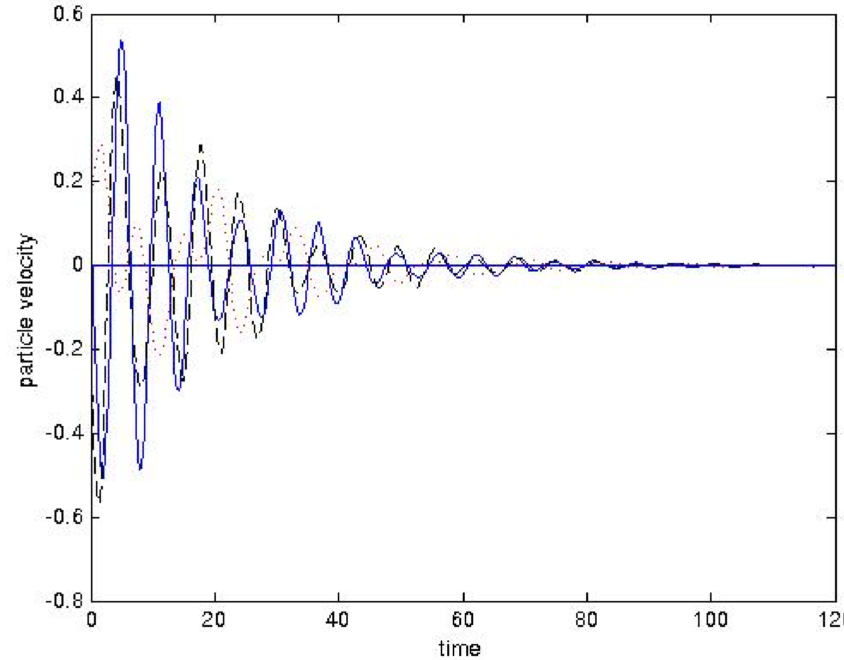
Matlab Simulation

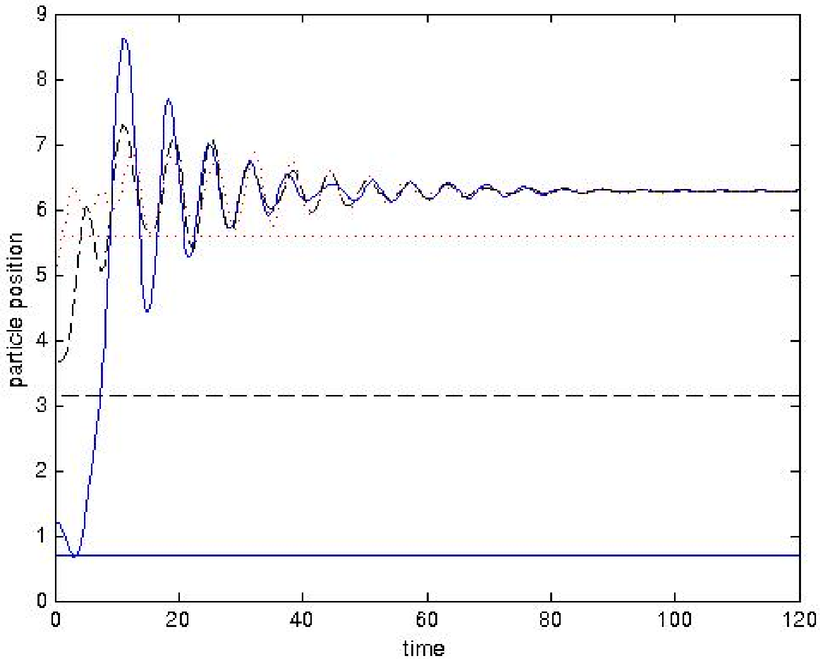


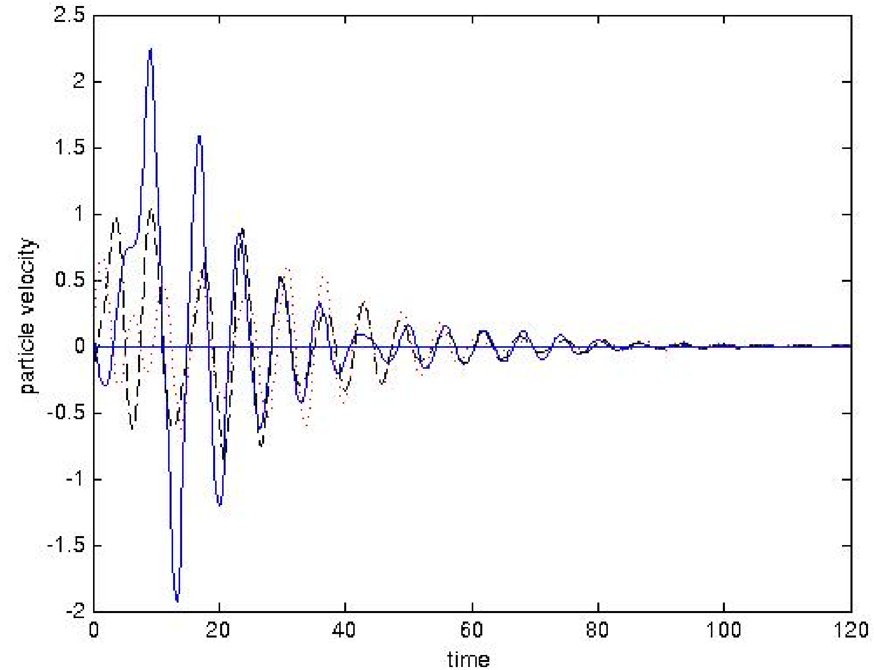


The first set of equilibrium points is stable

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.19, 0, 0.93, 0, 4.77, 0)$$

Matlab Simulation





The second set of equilibrium points is unstable

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.69, 0, 3.14, 0, 5.59, 0)$$

The Equilibrium Points

- In the case of nonlinear particle interaction, the equilibrium is at $(x_{i1},x_{i2})=(x_{i1}^*,0)$, where x_{i1}^* are solutions to

$$-\sin x_{11}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{21}^*-x_{11}^*)} - e^{-2\beta(x_{21}^*-x_{11}^*)}\right\} = 0,$$

$$-\sin x_{i1}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{i+1,1}^*-x_{i1}^*)} - e^{-2\beta(x_{i+1,1}^*-x_{i1}^*)}\right\}$$

$$-\frac{\kappa}{\beta}\left\{e^{-\beta(x_{i1}^*-x_{i-1,1}^*)} - e^{-2\beta(x_{i1}^*-x_{i-1,1^*})}\right\} = 0, i = 2,\ldots,N-1,$$

$$-\sin x_{N1}^* - \frac{\kappa}{\beta}\left\{e^{-\beta(x_{N1}^*-x_{N-1,1}^*)} - e^{-2\beta(x_{N1}^*-x_{N-1,1}^*)}\right\} = 0$$

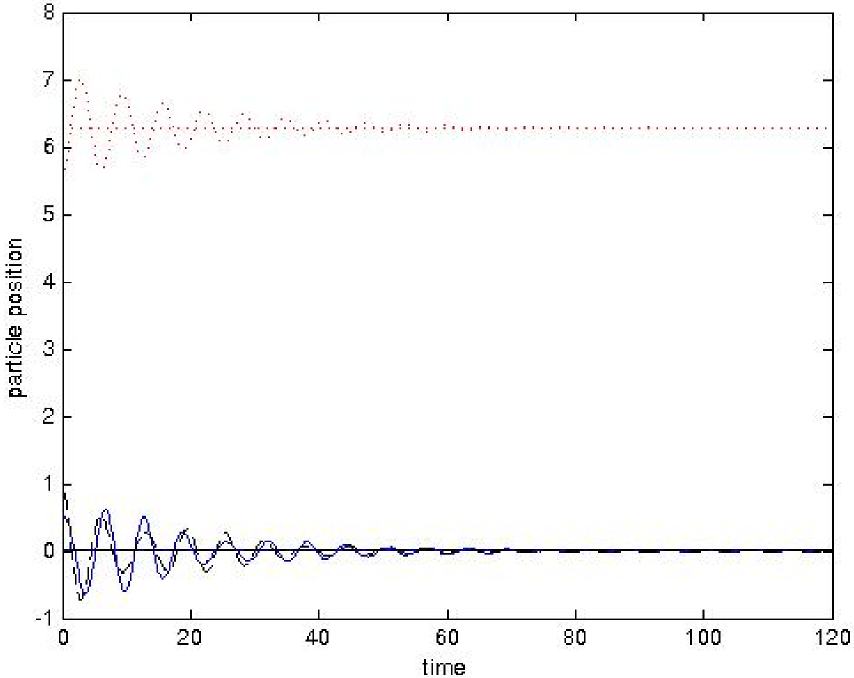
- An example system: N=3, κ=0.26, γ=0.1,β=1
- The system may have infinite number of equilibrium points
- Two sets of equilibrium points are at:
  - Set 1:

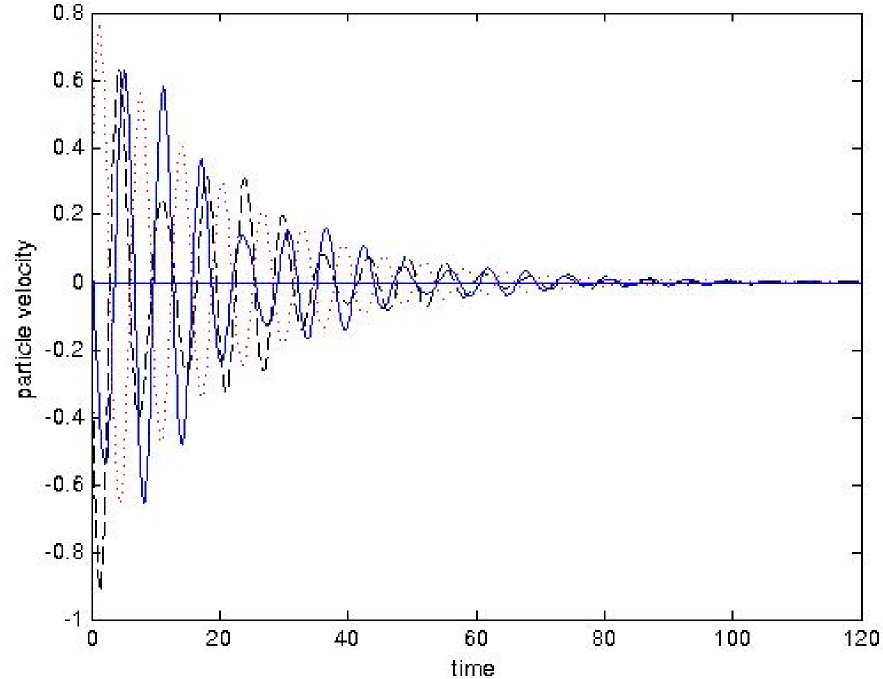
$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.0001, 0, 0.0004, 0, 6.28, 0)$$

  - Set 2:

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.01, 0, 3.14, 0, 6.27, 0)$$

Matlab Simulation

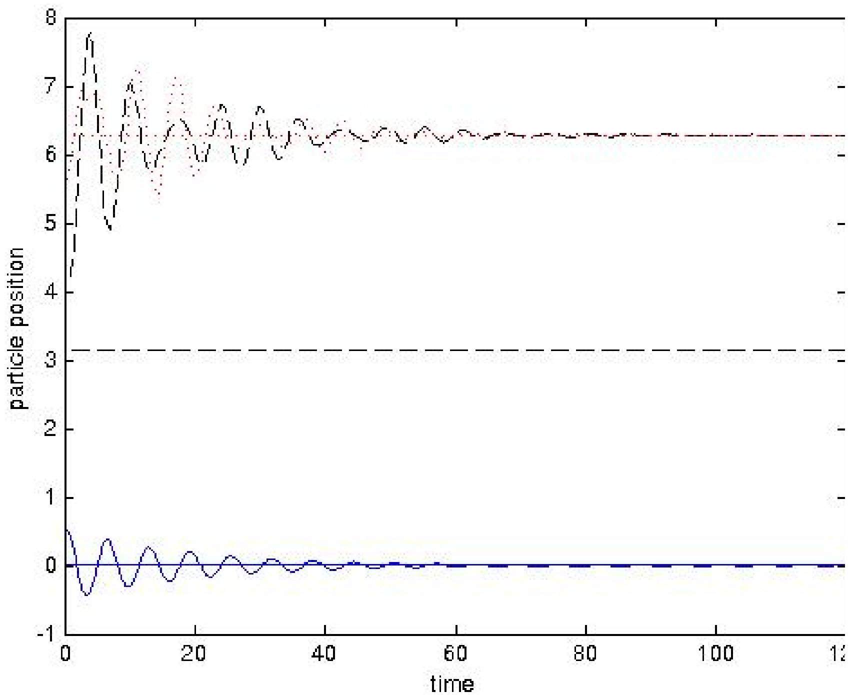


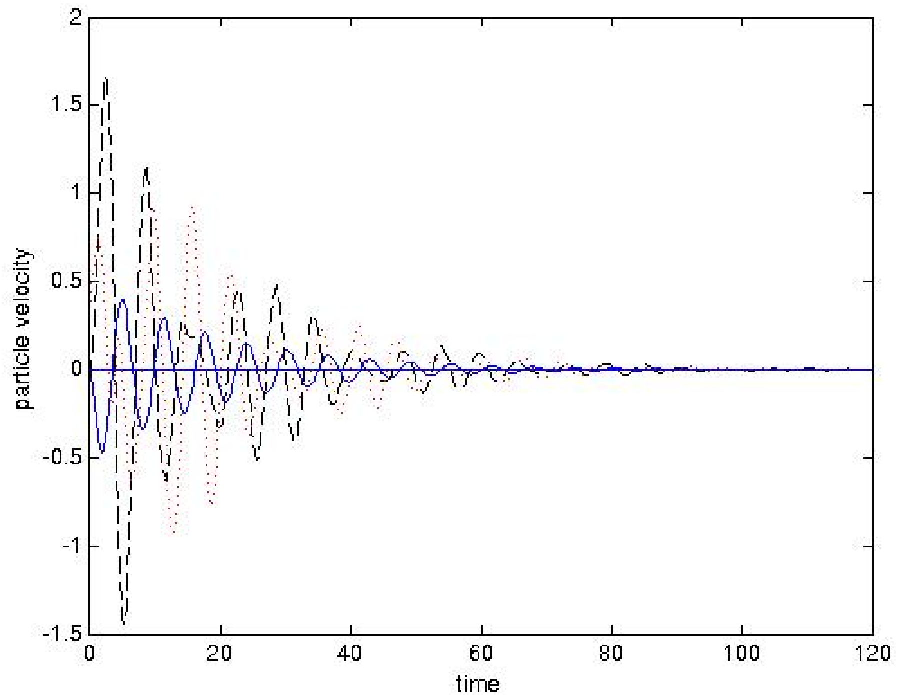


The first set of equilibrium points is stable

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.0001, 0, 0.0004, 0, 6.28, 0)$$

Matlab Simulation





The second set of equilibrium points is unstable

$$(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.01, 0, 3.14, 0, 6.27, 0)$$

Given a set of equilibrium points, how to tell its stability without doing simulations?

Open-Loop Stability with Linear Particle Interaction

- Linearize around equilibrium (x_{i1}^*,0), and define new states $z_{i1}=x_{i1}-x_{i1}^*$, $z_{i2}=x_{i2}$ we have

$$
\begin{aligned}
\dot{z}_{i1} &= z_{i2} \\
\dot{z}_{i2} &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1} - 2z_{i1} + z_{i-1,1}) \\
& \quad \boxed{-\sin x_{i1}^* + \kappa(x_{i+1,1}^* - 2x_{i1}^* + x_{i-1,1}^*)} \quad =0 \\
&= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \kappa(z_{i+1,1} - 2z_{i1} + z_{i-1,1})
\end{aligned}
$$

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- Stacking the equations for i=1,2,…,N:

$$\dot{z} \quad = \quad Az + BFz$$

$$A = I_N \otimes A_i, B = I_N \otimes B_i, F = Q \otimes \left[\begin{array}{cc} 1 & 0 \end{array}\right],$$

$$A_i = \left[\begin{array}{cc} 0 & 1 \\ 0 & -\gamma \end{array}\right], B_i = \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

$$Q = \left[\begin{array}{ccccc} -\kappa - \cos x^*_{11} & \kappa & 0 & \dots & 0 \\ \kappa & -2\kappa - \cos x^*_{21} & \kappa & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & \kappa & -2\kappa - \cos x^*_{N-1,1} & \kappa \\ 0 & \dots & 0 & \kappa & -\kappa - \cos x^*_{N1} \end{array}\right]$$

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Open-Loop Stability with Linear Particle Interaction

- Theorem 1:

  The open-loop system with linear particle interaction is locally asymptotically stable at the equilibrium points $(x_{i1}^*, 0)$ if all of the eigenvalues of the matrix Q have negative real parts; it is unstable if any of the eigenvalues of the matrix Q has a positive real part.

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Open-Loop Stability with Linear Particle Interaction

- Particularly, we have the following cases:

1. If $\cos x_{i1}^* \geq 0$ for all i with strict sign for at least one i, Q is Hurwitz and the system is asymptotically stable;
2. If $\cos x_{i1}^* = 0$ for all i, Q has one (and only one) eigenvalue 0. The linearized system is marginally stable and the stability of the original nonlinear system could be either stable or unstable;
3. If $\cos x_{i1}^* \leq 0$ for all i with strict sign for at least one i, Q has at least one positive eigenvalue. The system is unstable;
4. If $\cos x_{i1}^*, i = 1, \ldots, N$ have mixed signs, the system could be either stable or unstable and numerical calculations is necessary to determine the sign of the real parts of the eigenvalues of Q.

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Open-Loop Stability with Linear Particle Interaction

- Special cases:
  - $2k\pi$ are stable equilibrium
  - $(2k+1)\pi$ are unstable equilibrium

Y. Guo, Z. Qu, and Z. Zhang, "Lyapunov stability and precise control of the frictional dynamics of a one-dimensional particle array", Physical Review B, Vol. 73, No. 9, 2006.

- Outline of proof:

Define a similarity transformation $z = \overline{T}\zeta$. In the new coordinate, the system dynamics is $\dot{\zeta} = H\zeta$.

Since Q is a real symmetric matrix, there exists a unitary matrix T such that $T^{-1}QT = D$ where D is a diagonal matrix of eigenvalues of Q.

Let the transformation matrix be

$$\overline{T} \quad = \quad T \otimes I_2 \tag{1}$$

where I_2 is the 2×2 identity matrix. We can obtain $H = \mathrm{d}iagH_{ii}$ and

$$H_{ii} \quad = \quad \begin{bmatrix} 0 & 1 \\ \alpha_i & -\gamma \end{bmatrix},$$

where $\alpha_i, i = 1, 2, \ldots, N$ are eigenvalues of Q.

The sign of α_i determines whether the eigenvalues of H_{ii} have negative real parts at the equilibrium point.

Open-Loop Stability with Linear Particle Interaction

- Checking the two simulation examples:
  - Set 1: $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.19, 0, 0.93, 0, 4.77, 0)$ cos(0.19)>,cos(0.93)>0,cos(4.77)>0, case 1, asymptotically stable;
  - Set 2: $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.69, 0, 3.14, 0, 5.59, 0)$ cos(0.69)>0,cos(3.14)<0,cos(5.59)>0, case 4, check eigenvalues of Q, ➡unstable system.

Q =

-1.0312	0.2600	0
0.2600	0.4800	0.2600
0	0.2600	-1.0292

eig(Q) =

-1.1150
-1.0302
0.5648

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Open-Loop Stability with Nonlinear Particle Interaction

- Recall that the equilibrium points are at $(x_{i1},x_{i2})=(x_{i1}^*,0)$, where x_{i1}^* are solutions to

$$-\sin x_{11}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{21}^*-x_{11}^*)} - e^{-2\beta(x_{21}^*-x_{11}^*)}\right\} = 0,$$

$$-\sin x_{i1}^* + \frac{\kappa}{\beta}\left\{e^{-\beta(x_{i+1,1}^*-x_{i1}^*)} - e^{-2\beta(x_{i+1,1}^*-x_{i1}^*)}\right\}$$

$$-\frac{\kappa}{\beta}\left\{e^{-\beta(x_{i1}^*-x_{i-1,1}^*)} - e^{-2\beta(x_{i1}^*-x_{i-1,1^*})}\right\} = 0, i = 2, \ldots, N-1,$$

$$-\sin x_{N1}^* - \frac{\kappa}{\beta}\left\{e^{-\beta(x_{N1}^*-x_{N-1,1}^*)} - e^{-2\beta(x_{N1}^*-x_{N-1,1}^*)}\right\} = 0$$

- Linearize the system around its equilibrium $(x_{i1}^*, 0)$, and define new states $z_{i1}=x_{i1}-x_{i1}^*$, $z_{i2}=x_{i2}$ we have

$$
\begin{aligned}
\dot{z}_{i1} &= z_{i2} \\
\dot{z}_{i2} &= -\cos x_{i1}^* z_{i1} - \gamma z_{i2} \\
&+ \frac{\kappa}{\beta}\left[-e^{-\beta(x_{i+1,1}^* - x_{i1}^*)} + 2e^{-2\beta(x_{i+1,1}^* - x_{i1}^*)}\right](z_{i+1,1} - z_{i1}) \\
&- \frac{\kappa}{\beta}\left[-e^{-\beta(x_{i,1}^* - x_{i-1,1}^*)} + 2e^{-2\beta(x_{i1}^* - x_{i-1,1}^*)}\right](z_{i1} - z_{i-1,1}) \\
&\stackrel{def}{=} -\cos x_{i1}^* z_{i1} - \gamma z_{i2} + \boxed{c_{i1}}(z_{i+1,1} - z_{i1}) \\
&\qquad - \boxed{c_{i2}}(z_{i1} - z_{i-1,1})
\end{aligned}
$$

Coupling coefficients

- Notice the same structure as in the linear interaction case with different coupling coefficients

- Stacking the equations for i=1,2,…,N:

$$\dot{z} \quad = \quad Az + BFz$$

$$A = I_N \otimes A_i, B = I_N \otimes B_i, F = Q \otimes \left[\begin{array}{cc} 1 & 0 \end{array} \right],$$

$$A_i = \left[\begin{array}{cc} 0 & 1 \\ 0 & -\gamma \end{array} \right], B_i = \left[\begin{array}{c} 0 \\ 1 \end{array} \right],$$

$$Q = \left[\begin{array}{ccccc} -c_{11} - \cos x^*_{11} & c_{11} & 0 & \dots & 0 \\ c_{21} & -(c_{21} + c_{22} + \cos x^*_{21}) & c_{22} & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & c_{N-1,1} & -(c_{N-1,1} + c_{N-1,2} + \cos x^*_{N-1,1}) & c_{N-1,2} \\ 0 & \dots & 0 & c_{N2} & -c_{N2} - \cos x^*_{N1} \end{array} \right].$$

Open-Loop Stability with Nonlinear Particle Interaction

- Theorem 2:

  The stability of the nonlinear system is locally asymptotically stable at the equilibrium points $(x_{i1}^{*},0)$ if all of the eigenvalues of the matrix Q have negative real parts; it is unstable if any of the eigenvalues of the matrix Q has a positive real part.

Open-Loop Stability with Nonlinear Particle Interaction

- Checking the two simulation examples:
  - Set 1: $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.0001, 0, 0.0004, 0, 6.28, 0)$

Q =

-1.2598	0.2598	0
-0.0005	-1.2593	0.2598
0	0.5195	-1.5195

eig(Q) =

-1.2595
-1.0000
-1.7790

$\Rightarrow$ asymptotically stable;

  - Set 2: $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, \phi_3, \dot{\phi}_3) = (0.01, 0, 3.14, 0, 6.27, 0)$

Q =

-0.9896	-0.0104	0
-0.0104	1.0207	-0.0104
0	0.5086	-1.5085

eig(Q) =

-0.9896
1.0187
-1.5065

$\Rightarrow$ unstable system.

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Tracking Control Design

- Design feedback control u(t) such that $\dot{\phi}_i$ tracks a constant targeted velocity, v_{target}

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right) + u(t)$$

Tracking Control Design

- Define average error states:

$$e_{1av} = \phi_{cm} - v_{target}t, e_{2av} = v_{cm} - v_{target},$$

- State-space model:

$$\begin{aligned}
\dot{e}_{1av} &= e_{2av} \\
\dot{e}_{2av} &= -\frac{1}{N}\sum_{i=1}^{N}\sin(e_{i1} + v_{target}t) - \gamma(e_{2av} + v_{target}) \\
&\quad + u(t)
\end{aligned}$$

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Tracking Control Design

- Theorem 3:

  The following feedback control law renders the error states of the closed-loop system bounded:

$$
\begin{aligned}
u(t) &= \gamma v_{target} - e_{1av} - (c_1 - \gamma) e_{2av} \\
&\quad -(c_1 + c_2)\xi + \sin(v_{target} t) \\
&= \gamma v_{target} - k_1 (\phi_{cm} - v_{target} t) \\
&\quad -k_2 \left(v_{cm} - v_{target}\right) + \sin(v_{target} t)
\end{aligned}
$$

Average quantities

Tracking Control Design

- Outline of proof:
  - Choose Lyapunov function candidate:

$$W = \frac{1}{2}e_{1av}^2 + \frac{1}{2}(c_1 e_{1av} + e_{2av})^2$$

  - Along the closed-loop dynamics, we have:

$$\dot{W} \leq -c_1(e_{1av}^2 + \xi^2) + \frac{1}{c_2}$$

where where $\xi = c_1 e_{1av} + e_{2av}$

Tracking Control Design

– We obtain:

$$\dot{W}(e_{av}) \leq 0, \qquad \forall \|(e_{1av}, \xi)\| \geq \frac{1}{\sqrt{c_1 c_2}}$$

– The ultimate bound of $||e_{av}||$ is:

$$b = \sqrt{\frac{\lambda_{max}(P)}{c_1 c_2 \lambda^2_{min}(P)}} \quad where \quad P = \begin{bmatrix} 1 + c_1^2 & c_1 \\ c_1 & 1 \end{bmatrix}$$

– This indicates that by choosing c_1, c_2 appropriately, the error states can be made arbitrarily close to zero.

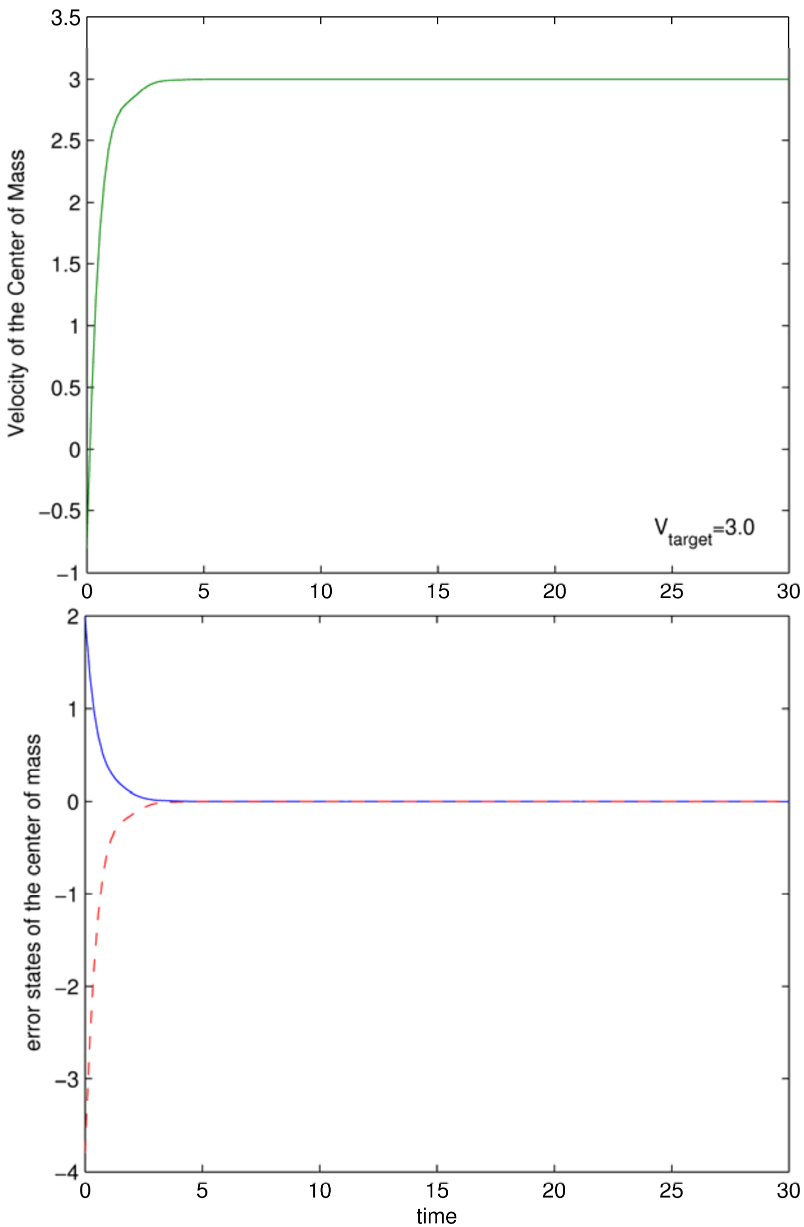
Tracking Control Design

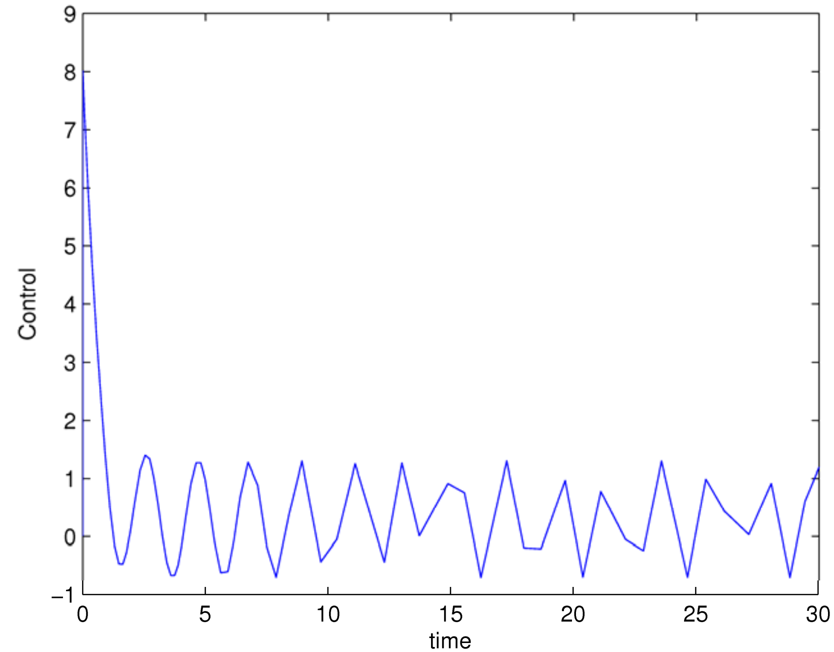
- To achieve asymptotically tracking, the following switching-type control law can be used:

$$
\begin{aligned}
u(t) &= \gamma v_{target} - k_1(\phi_{cm} - v_{target} t) \\
&\quad -k_2\left(v_{cm} - v_{target}\right) + \sin(v_{target} t) - 2sgn(\xi)
\end{aligned}
$$

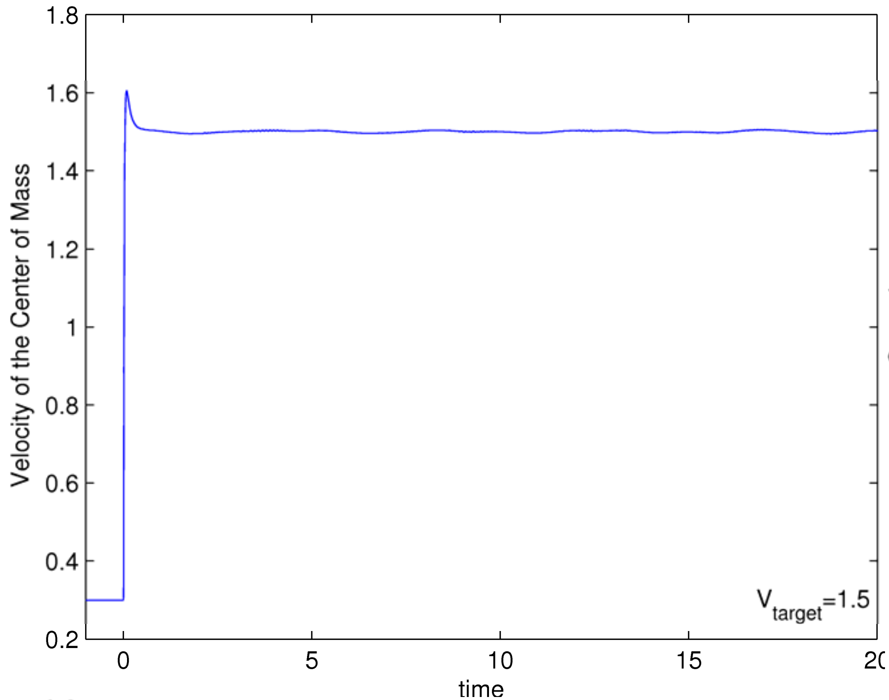
switching

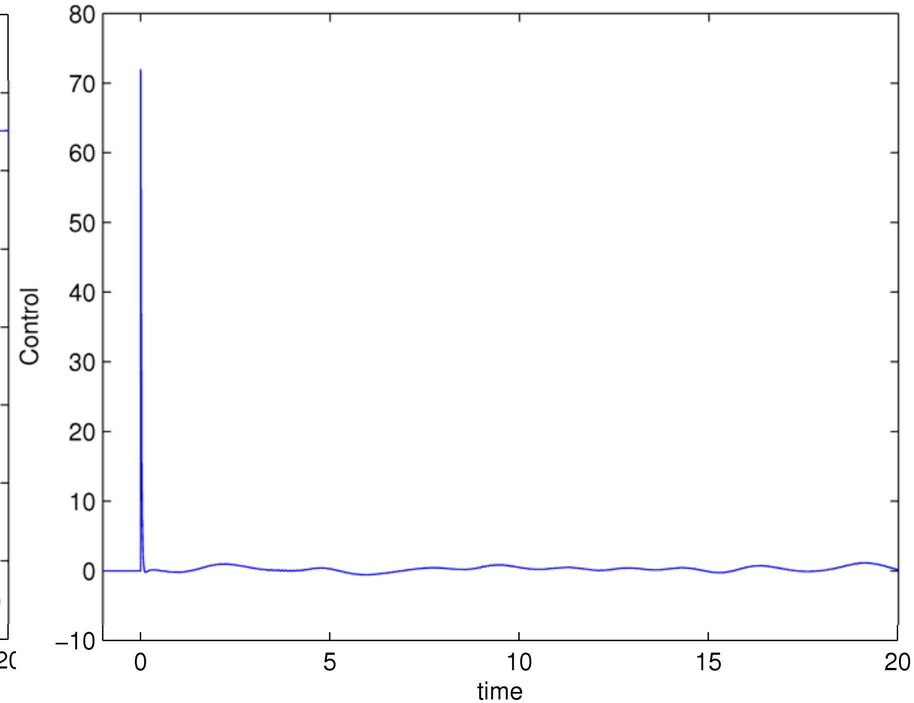
Matlab Simulation

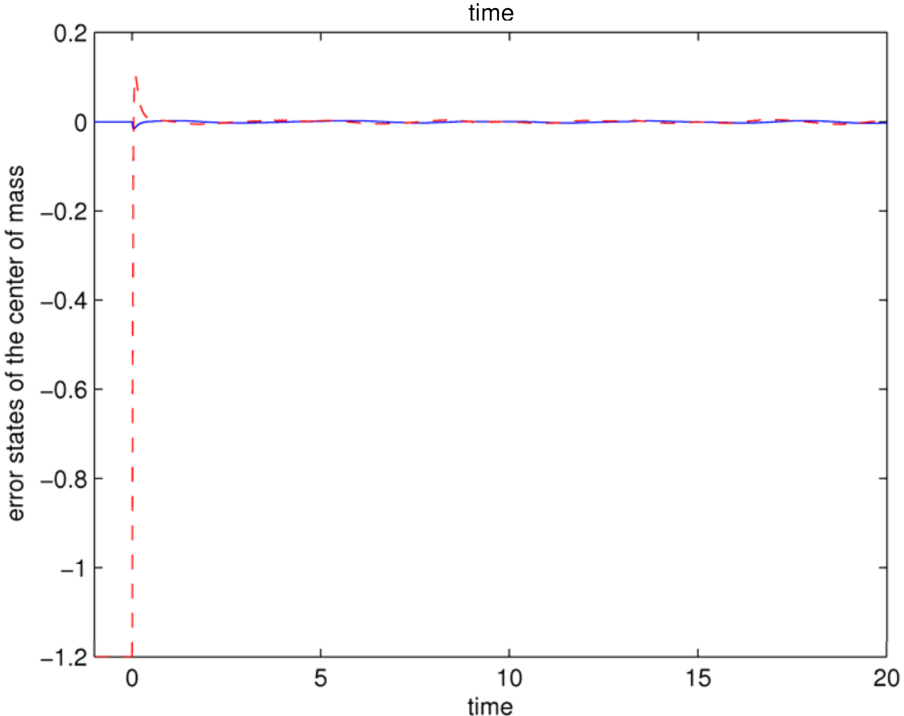




Tracking control performance for targeted velocity v_{target}=3







Tracking control performance for targeted velocity v_{target}=1.5

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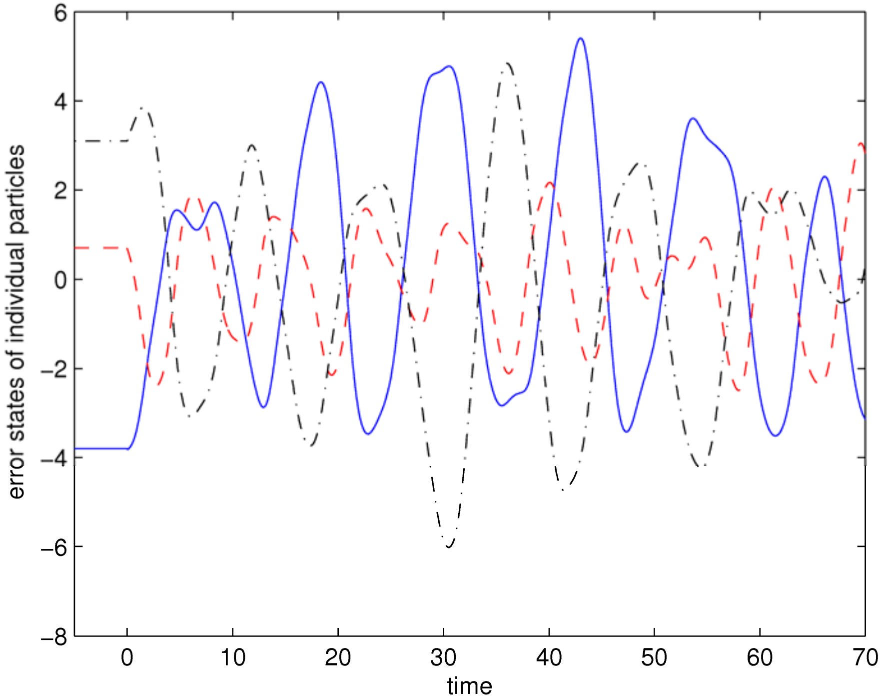
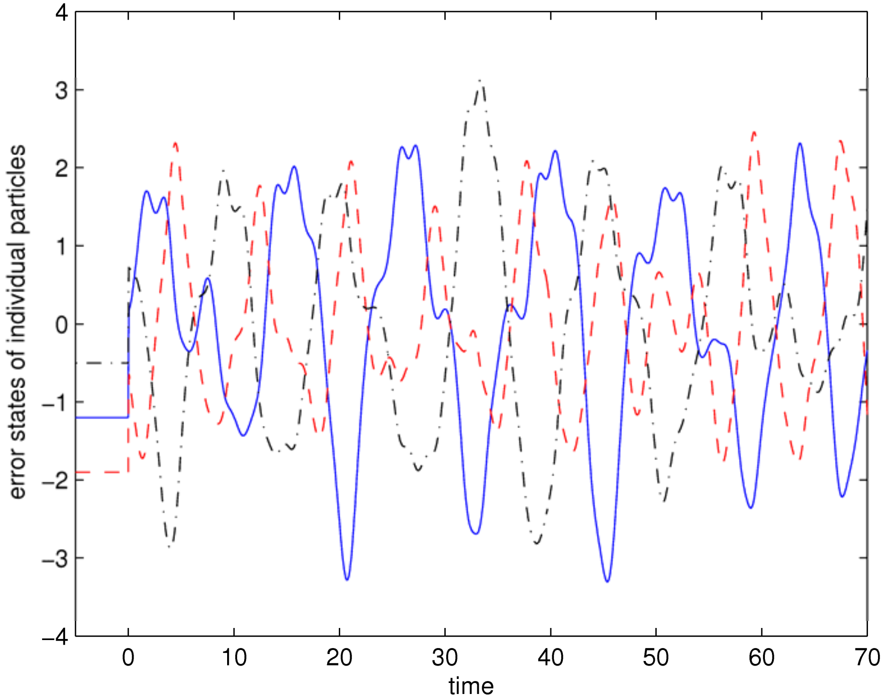
However, individual particles are not necessarily stable in the closed-loop system under average control!!

Single Particle Dynamics



The phase variable of individual particles



The velocity variable of individual particles

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Outline

- Motivation
- The Model and Control Problem Formulation
- Open-Loop Stability
- Tracking Control Design
- Single Particle Dynamics
- Conclusion

Stability of Single Particles in the Closed-Loop System

- Define error states of individual particles:

$$e_{i1} = \phi_i - v_{target} t, \qquad e_{i2} = \dot{\phi}_i - v_{target}$$

- Representing error dynamics:

$$\begin{aligned}
\dot{e}_{i1} &= e_{i2} \\
\dot{e}_{i2} &= -\gamma e_{i2} + \kappa \left(e_{i+1,1} - 2e_{i1} + e_{i-1,1}\right) - \bar{k}_1 \left(\sum_{i=1}^{N} e_{i1}\right) - \bar{k}_2 \left(\sum_{i=1}^{N} e_{i2}\right) \\
&\quad + [\sin(v_{target} t) - \sin(e_{i1} + v_{target} t)]
\end{aligned}$$

Average control

- Theorem 4:

For system parameters γ and κ that satisfy

$$\kappa > \frac{1}{\min_{i \le N-1}(\mu_i)},$$
$$\gamma > \frac{1}{\sqrt{\min_{i \le N-1}(\mu_i)\kappa - 1}}$$

where $\mu_i, i = 1, \ldots, N-1$ are the positive eigenvalues of the matrix $(-Q)$, the average control asymptotically stabilize the error system if k_1 and k_2 are chosen to satisfy

$$k_1 \ge \kappa \min_{i \le N-1}(\mu_i), \qquad k_2 \ge 0.$$

Single Particle Dynamics

- This indicates that under certain conditions on system parameters (γ, κ), single particles can be stabilized under the average control, *i.e.*, the error system of individual particles is asymptotically stable.

- Outline of proof:

We re-present the error system in the following form:

$$\dot{E} = GE + f(e,t)$$

We show that under the transformation matrix $T = V \otimes I_2$, we have

$$T^{-1}GT = \mathrm{d}iagC_i.$$

Using the Lyapunov function

$$W(t,e) = E^T HE = E^T TPT^{-1}E = E^T(I_N \otimes P_i)E$$

we obtain

$$\dot{W}(t,e) = E^T(\mathrm{d}iagS_i)E$$

We showed that under conditions on γ, κ, the stability margin of the linear part of the system dominates the nonlinear part.

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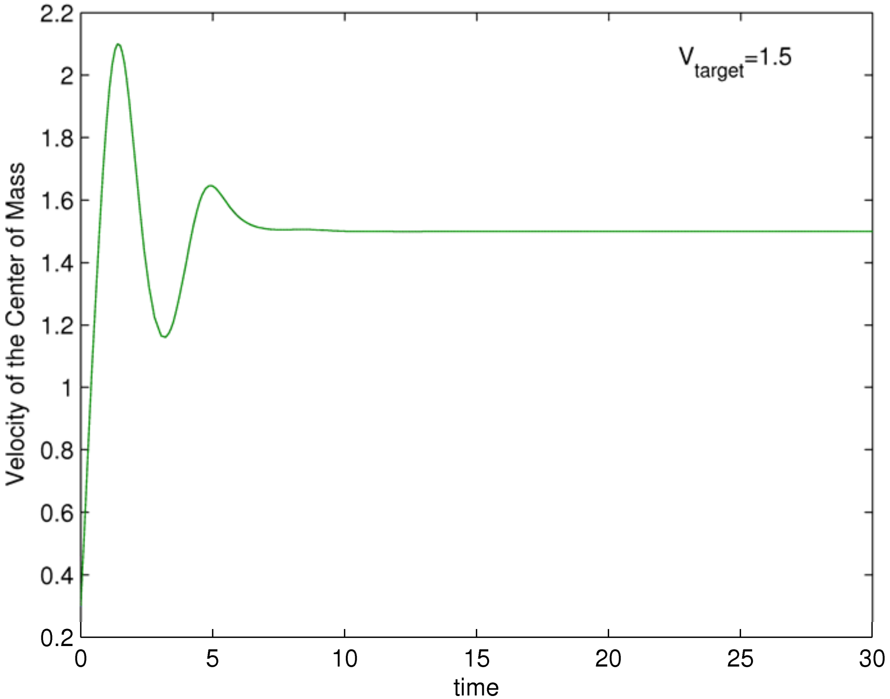
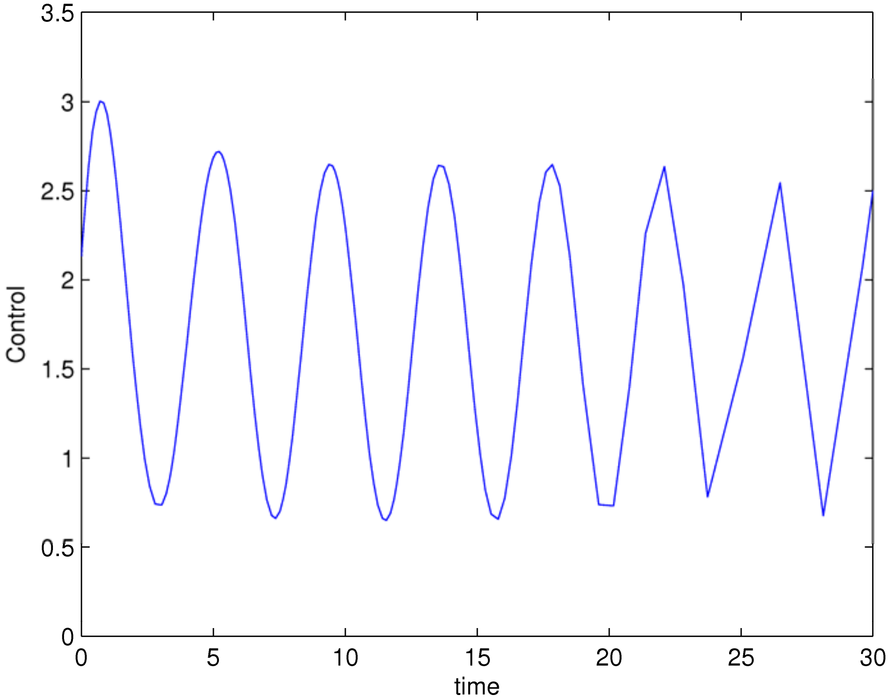
Simulation Results



(a)

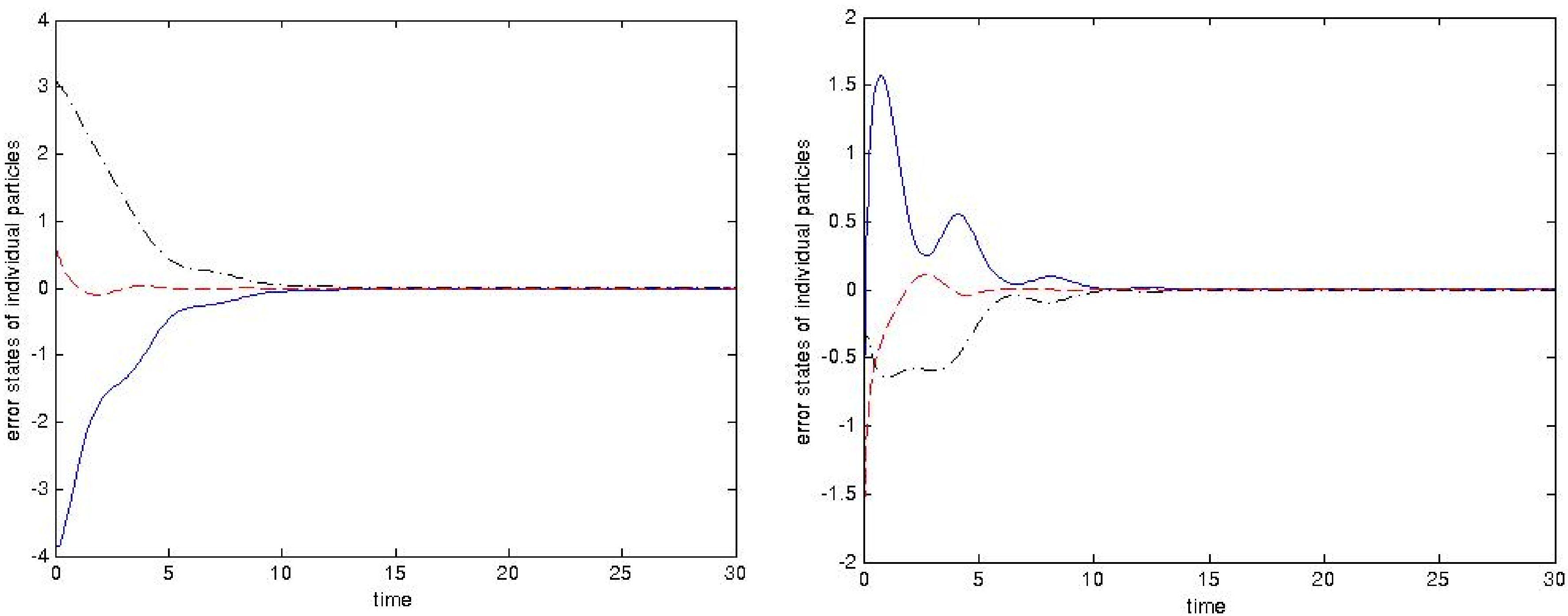


(b)

Tracking control of the average system

(a) the velocity of the center of mass, (b) the control.

Simulation Results



(a) (b)

Error states for individual particles in the closed-loop system (a) the phase variables, (b) the velocity variables.

Conclusions

- Motivated by friction control at the nanoscale, we considered the stability and tracking control of the nonlinear interconnected system represented by the FK-model
- Control theoretical methods are used to analyze the stability of open-loop system, and to design tracking control law utilizing average quantities only
- Matlab simulations verified theoretical results

Future Research (Other Applications)

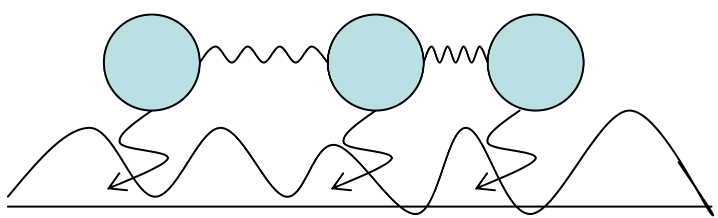
- Directed motion is induced by breaking the symmetry of particle interactions ➡ molecular car

$$m\ddot{x}_i + \gamma \dot{x}_i = -\frac{\partial U(x_i)}{\partial x_i} - \frac{\partial W(x_i - x_j)}{\partial x_i} + f_i + \eta(t)$$

FK model

$$W(x_i - x_j) = \frac{\kappa}{2}[\|x_i - x_j\| - a_{ij}(t)]^2,$$

Nonsymmetric interactions



M. Porto, M. Urbakh, and J. Klafter. Atomic scale engines: Cars and wheels. Physical Review Letters, 84(26):6058–6061, 2000.

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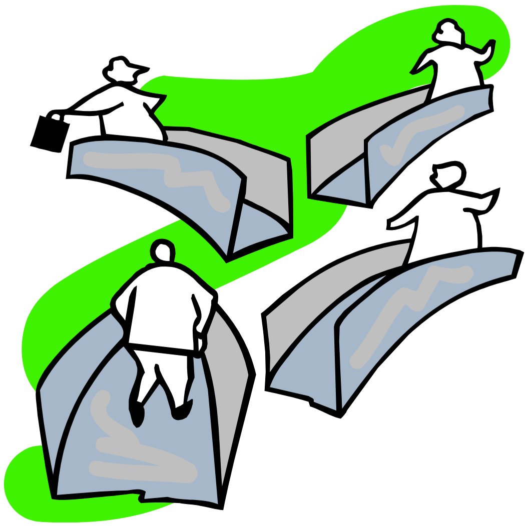
Future Research

- Control theoretical methods can be applied to generate analytic results for precise control of the “molecular car”
- But the real challenge is how to implement it?
- A “molecular highway” in the future?

Thank you for listening!



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