From Trajectories to the Ergodic Partition

An Algorithm

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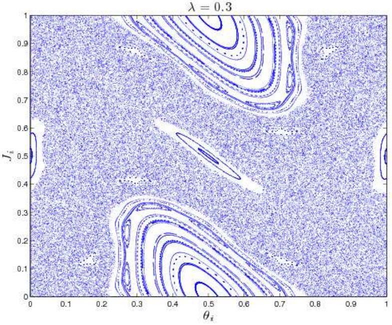
Dynamics Days, Jan 3–6, 2008
Knoxville, TN

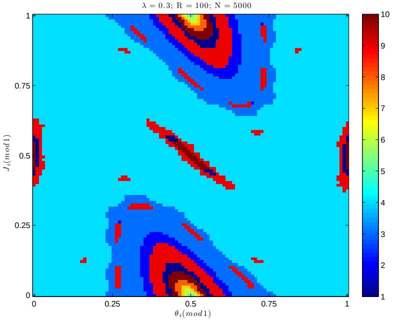
Motivation and purpose

Trajectory plot



$\Rightarrow$

Approx. of ergodic partition



Target

Measure-preserving map on a finite/periodic domain.

Goal

Quick, coarse partition of phase space.

15 min talk in one slide

Core idea

Ergodic subsets – dynamical *atoms* in phase space.

How can we do it?

"Concatenate" trajectories using data clustering methods.

Why do we care?

Analysis – mapping out phase space

Design – easier to exploit natural dynamics of system

Does our solution measure up?

Yes.

Fast ($\sim$ minutes) two-step algorithm.

Partition corresponds to dynamics in known problems.

Theoretical framework

Birkhoff's ergodic theorem

For system $T : \mathcal{M} \to \mathcal{M}$, if $\mathcal{X} \subset \mathcal{M}$ is an *ergodic subset*, then for $\forall x \in \mathcal{X}, \forall f \in L^1_\mu(\mathcal{M})$

$$\overline{f} = \overbrace{\int_{\mathcal{X}} f(x) d\mu(x)}^{\text{spatial average}} = \overbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} f\left[T^k(x)\right]}^{\text{temporal average}} = f^*(x),$$

- Level sets of $f^* \longrightarrow$ *invariant* partition $\mathcal{P}_f$
- Ergodic partition $\mathcal{P}_E := \bigvee_{f \in L^1_\mu} \mathcal{P}_f$

Grouping criterion

$$f^*(x) = f^*(y), \forall f \in L^1_\mu$$
$$\Downarrow$$
$$x, y \in \mathcal{X}$$

On a conceptual level

Algorithm:

1. Choose a *good basis* for observables,
2. Pick a *large* number of ICs in phase space,
3. Simulate system from each IC for an *infinite* time,
4. Compute time averages of observables along trajectories,
5. *Group* trajectories with same time averages into sets.

Limitations:

- No basis in general for $L^1_\mu(\mathcal{M})$, countably infinite for $L^2_\mu(\mathcal{M})$,
- Only finite density of ICs can be chosen,
- Only finite time evolution is computable.

Result: Implementation of grouping criterion unclear.

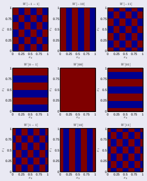
Implementation

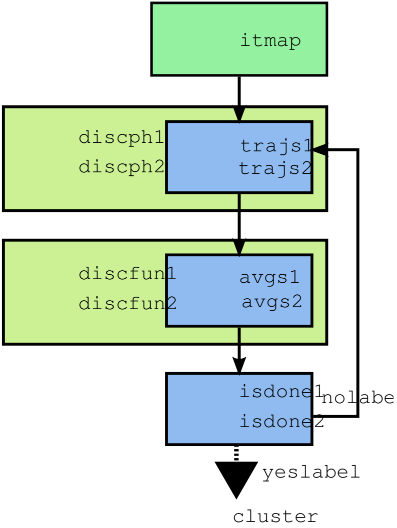
Step 1/2: Simulation

Purpose

Associate time average vector v_i with every trajectory.

Periodic Haar basis





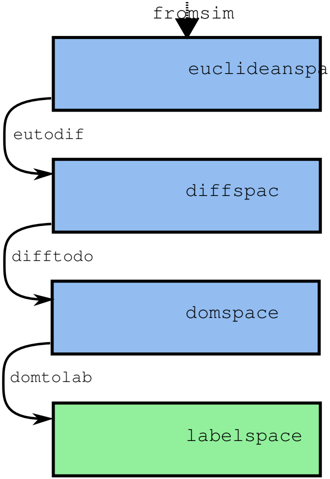
Implementation

Step 2/2: Clustering

Purpose

Implement grouping criterion – assign the same label to similar trajectories.

1. Euclidean graph:
   nodes $\longrightarrow$ trajectories
   edges $\longrightarrow g_{ij} = ||v_i - v_j||_2$
2. Diffusion distance graph: adds robustness to data distribution
3. Dominant eigenspace of random walk: natural coordinate system
4. CC Clustering: reveals and labels dominant features



Test system

Standard map

Description

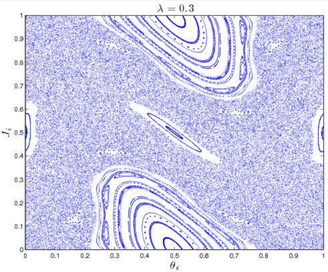
$$J_{n+1} = J_n + \lambda \sin(2\pi\theta_n) \qquad \pmod 1$$

$$\theta_{n+1} = J_{n+1} + \theta_n \qquad \pmod 1$$

$$f : S^2 \to \mathbb{R}^n \qquad f \in L^2(S^2)$$

- Poincaré map of periodically forced harmonic oscillator
- Measure-preserving; resonant and chaotic zones
- $\lambda \in (0, 1)$ tunes amount of chaos
- Observables – Haar basis on S^2

Trajectory plot

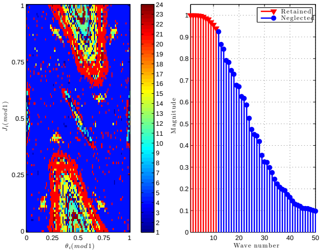


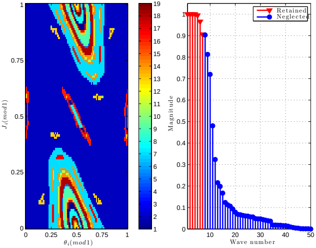
Unnecessary detail in chaotic region. No obvious way of color-coding regions.

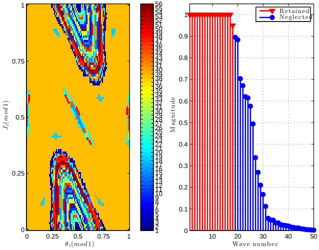
Partition quality analysis

Averaging horizon length

- $\lambda = 0.3$; $100[Box/Dim]$
- Iterations (clockwise): 50, 600, 1700
- More iterations:
    - Longer simulation step
    - High spectral ridge (gap)



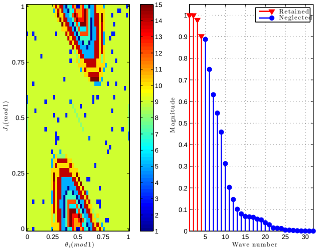


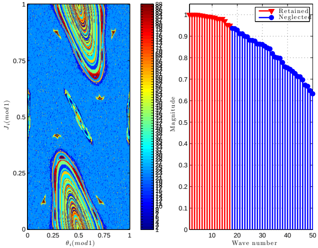


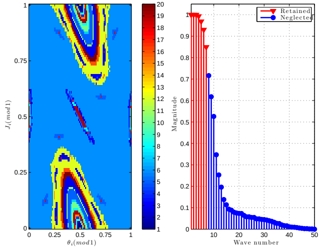
Partition quality analysis

Discretization resolution

- $\lambda = 0.3$; 2000 Iterations
- Resolution $[Box/Dim]$ (clockwise): 50, 106, 300
- Higher resolution:
    - Longer clustering step
    - Low spectral ridge (gap)



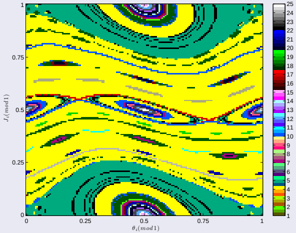




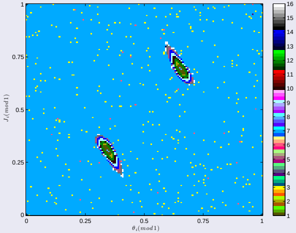
Running time analysis

Complexity of dynamics

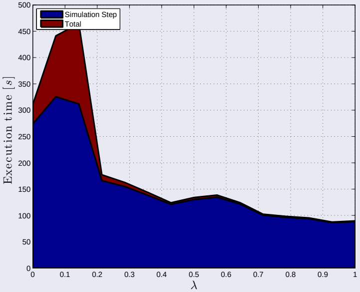
$\lambda = 0.10$



$\lambda = 0.75$



- Blue – simulation step
- Red – total running time



Influence of dynamics (λ)

Running time analysis

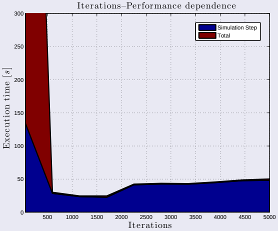
Algorithm parameters

More iterations:

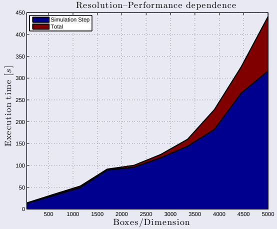
- asymptotical behavior more faithfully approximated,
- increased simulation cost,
- reduced clustering cost.

Higher resolution:

- able to resolve finer features,
- increased simulation cost,
- increased clustering cost.



Influence of no. of iterations



Influence of resolution

Final remarks

Achievements

Goal valid partitioning

Speed order of minutes on a laptop

Efficiency only interesting segments of phase space can be analyzed

Challenges

- Tuning of clustering step
- Validation of result for unknown behavior
- Parallelization – necessity for higher dimensions
- Extension to continuous time systems (ODEs/DAEs)

Conclusion

Efficient computational tool improvable with further theoretical development.

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